Math 868 — Homework 8

Due Monday, Nov 5

These problems relate to Chapter 11 in Lee's textbook. All vector spaces are real vector spaces.

1. Prove that $V \otimes \mathbf{R} \cong V$ as follows.
   (a) Define $A : V \oplus \mathbf{R} \to V$ by $A(v, c) = cv$. Prove that A is bilinear.
   (b) Apply the universal property and show that the resulting map $\tilde{A}$ is an isomorphism.

2. (a) If V and W are vector spaces of finite positive dimensions n and m, prove that the set $L(V, W)$ of all linear maps $\lambda : V \to W$ is a vector space of dimension mn.
   (b) Prove that $L(V, W) \cong V^* \otimes W$. (*Hint:* define $A : V^* \oplus W \to L(V, W)$ by $A(\alpha, w)(v) = \alpha(w)\, v$. As in Problem 1, show this is bilinear, apply the universal property, and show the resulting map $\tilde{A}$ is an isomorphism.

3. (a) Let V be an n-dimensional real vector space. Show that

$$\dim \mathrm{Sym}^k(V) \;=\; \binom{n+k-1}{n-1} \;=\; \frac{(n+k-1)!}{k!(n-1)!}$$

   Hint: This is the number of degree k monomials in n variables. Draw $n + k - 1$ dots in a row, put slashes through $n - 1$ of them, and use these picture to assign powers to the variables $x_1, \ldots, x_n$.
   (b) What is the dimension of $\mathrm{Sym}^2(V)$? How many degree 5 monomials are there in the variables x, y, z?

4. (a) Do Problem 12-2 in Lee, page 319.
   (b) Do Problem 12-5 in Lee, page 320.

5. Let M be a n-dimensional manifold. Prove that n functions $f^1, \ldots, f^n : M \to \mathbf{R}$ forms a coordinate system in a neighborhood of $p \in M$ if and only if $df^1 \wedge \cdots \wedge df^n \neq 0$ in $\Lambda^n(T_p^* M)$.