Math 868 — Homework 7

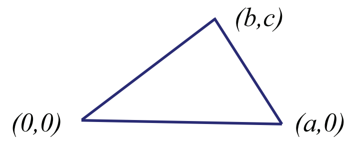
Due Wednesday, Oct 24

1. Evaluate $\displaystyle\int_C \omega$ where $c : [0,1] \to \mathbf{R}^3$ by $c(t) = (t, t^2, t^3)$ and $\omega = y\ dx + 2x\ dy + y\ dz$.

2. For $\alpha = \frac{1}{2}(x\ dy - y\ dx)$, find

   (a) $\displaystyle\int_{C_R} \alpha$ where C_R is the circle $\{x^2 + y^2 = R^2\}$ in $\mathbf{R}^2$ with counterclockwise orientation.

   (b) $\displaystyle\int_T \alpha$ where T is the oriented triangle in $\mathbf{R}^2$ shown.



3. Determine whether each form is exact. If it is, find all functions f such that $\omega = df$.

   (a) $\omega = xy\ dx + \frac{1}{2}x^2\ dy$ on $\mathbf{R}^2$.

   (b) $\omega = x\ dx + xz\ dy + xy\ dz$ on $\mathbf{R}^3$.

   (c) $\omega = y\ dx$ on $\mathbf{R}^2$.

   (d) $\omega = \left(\frac{1}{x^2} + \frac{1}{y^2}\right)(y\ dx - x\ dy)$ on $\{(x,y) \mid x \neq 0 \text{ and } y \neq 0\}$.

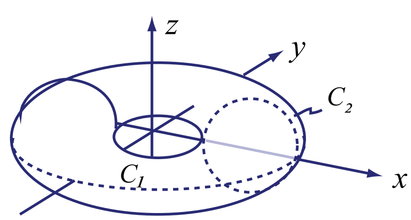
4. Let ω be the 1-form $\omega = \left(\frac{-y}{x^2+y^2}\right)\ dx + \left(\frac{x}{x^2+y^2}\right)\ dy$ on $\mathbf{R}^2 \setminus \{0\}$.

   (a) Calculate $\int_C \omega$ for any circle C around the origin.

   (b) Prove that in the half-plane $\{x > 0\}$, ω has the form df for some function f (try $\arctan(y/x)$ as a random possibility).

   (c) Prove that ω cannot be of the form dg on all of $\mathbf{R}^2 \setminus \{0\}$ for any function g.

5. Again let $\omega = \left(\frac{-y}{x^2+y^2}\right)\ dx + \left(\frac{x}{x^2+y^2}\right)\ dy$, now considered as a 1-form on $\mathbf{R}^3 \setminus \{z-\text{axis}\}$. Find $\int_{C_1} \omega$ and $\int_{C_2} \omega$ where C_1 and C_2 are the two curves shown on the torus of radius 1 around the core circle of radius 2 in the xy plane.



6. (Bonus) Write down a 1-form η on $\{(x,y,z) \mid z \neq 0 \text{ and } x^2 + y^2 \neq 2\}$ so that the numbers $\int_{C_1} \eta$ and $\int_{C_1} \eta$ are the same as integrals of ω in Problem 5 but in the opposite order. Verify by integrating.

Solutions

1. $\frac{34}{15}$.
2. In both (a) and (b) the integral is equal to the area enclosed by the path.
3. (a) and (d) are exact. Once you find f, you can check yourself that it works.
4. (a) 2π (follows from 2a). (b) Good guess! (c) Contradicts theorem that says ...
5. Apply 4a and 4b to get the answers without calculating.