Math 868 — Homework 5

Due Friday, Oct 12

1. Prove that the orthogonal group $O(n)$ is compact. (*Hint:* first show that if $A = (A_{ij})$ is orthogonal then $\sum_j A_{ij}^2 = 1$ for each i).

2. Verify that the tangent space to $O(n)$ at the identity matrix I is the vector space of all skew-symmetric $n \times n$ matrices, that is, the matrices A with $A^t = -A$. (Consider paths $B(t) = I + tA + \cdots$ in $O(n)$.)

3. The unitary group $U(n)$ is the group of all $n \times n$ complex matrices A such that $A^*A = Id.$, where $A^* = \overline{A}^t$ is the conjugate of the transpose (= transpose of the conjugate).

   (a) Prove that $U(n)$ is a Lie group (adapt the proof given in class for $O(n)$).

   (b) What is the tangent space to $U(n)$ at the identity matrix?

4. If G is a Lie group then, for each $g \in G$, left multiplication by g is a map $L_g : G \to G$ by $L_g h = gh$; this is smooth by the definition of Lie group.

   (a) Show that $L_g L_h = L_{gh}$ and that L_g is a diffeomorphism.

   (b) Prove that a group G that is locally a Lie group near $I \in G$ is a manifold ("locally a Lie group near $I \in G$" means there is a neighborhood U of I that is diffeomorphic to an open set in $\mathbf{R}^n$ and so that maps $U \to U$ by $g \mapsto g^{-1}$ and $g \to hg$ for h near I are smooth). (*Hint:* if $\phi : U \to G$ is a chart from an open set $U \subset \mathbf{R}^n$ with $\phi(0) = I$, then $L_g \circ \phi$ is a chart at g. Don't forget to show that the transition maps are diffeomorphisms).

   (c) Use (b) to give a different proof that $U(n)$ is a manifold.

5. Let $V = \{ai + bj + ck \in \mathbf{H}\}$ be the vector space of pure imaginary quaternions. As in class, each unit quaternion g gives a linear map $Ad_g : V \to V$ by $Ad_g(x) = gxg^{-1}$. If $g = \alpha i + \beta j + \gamma k$, write down Ad_g as a 3×3 matrix.

Solutions

1. Identify $Mat_n = \{n \times n \text{ real matrices}\}$ with $\mathbf{R}^{n^2}$ and define $\Phi : Mat_n \to Mat_n$ by $\phi(A_{ij}) = (A^T A)_{ij} = \sum_j A_{ij} A_{ij}$. Then Φ is continuous (it is a quadratic polynomial!), so $O(n) = \Phi^{-1}(Id.)$ is closed. Furthermore, for each $A \in O(n)$, we have $\sum_j A_{ij}^2 = 1$, so $\| A \|^2 = \sum_{ij} A_{ij}^2 = n$. Thus $O(n)$ is closed and bounded in $\mathbf{R}^{n^2}$, so compact.

2. If $A \in T_I O(n)$ there is a path $B(t)$ in $O(n)$ with $B(0) = I$ and $\dot{B}(0) = A$ satisfies $B^T(t)B(t) = I$. Applying $\frac{d}{dt}$ and evaluating at $t = 0$ gives $0 = \dot{B}^T(0) \cdot I + I \cdot \dot{B}^T(0) = A^T + A$, so A is skew-symmetric. Thus $T_I O(n) \subset \mathfrak{o}(n) = \{n \times n \text{ skew-symmetric matrices}\}$. Conversely, if $A \in \mathfrak{o}(n)$, then $B(t) = \exp(tA) = I + tA + \frac{1}{2}t^2 A^2 + \cdots$ satisfies $B(0) = I$, $\dot{B}(0) = A$ and

   $$B^T(t)B(t) = \exp(tA^T)\exp(tA) = \exp(-tA)\exp(tA) = I$$

   Consequently, $B(t)$ lies in $O(n)$ for all t and hence $A \in T_I O(n)$. Therefore $T_I O(n) = \mathfrak{o}(n)$.

3. (a) First, $U(n)$ is a group: if $A, B \in U(n)$ then (i) $(AB)^*(AB) = B^*A^*AB = B^*B = I$ and (ii)$A^*A = I \implies A^{-1} = A^* \implies (A^{-1})^* = A^{**} = A \implies (A^{-1})^* A^{-1} = I$. Thus $AB \in U(n)$ and $A^{-1} \in U(n)$.

   Let $H(n) = \{n \times n \text{ cx. matrices} \,|\, A^* = A\}$ be the vector space of hermitian matrices and define $\Phi : \mathbf{C}^{n^2} \to H(n)$ by $\Phi(A) = A^*A$. This is smooth (it is quadratic in the entries of A), $\Phi^{-1}(Id) = U(n)$, and the image lies in $H(n)$ since $(A^*A)^* = A^*A^{**} = A^*A$. One shows

$$d\Phi_A \text{ is onto}$$

exactly as was done in class for $O(n)$ with A^T replaced everywhere by A^*. Hence $U(n)$ is an immersed submanifold of $\mathbf{C}^{n^2}$. Finally, the group operations are smooth because they are restrictions of the smooth group operations of $GL(n,\mathbf{C})$ to the submanifold $U(n)$.

(b) Repeating Problem 2 above, with A^T replaced everywhere by A^*, shows that $T_I U(n)$ is the space $\mathfrak{u}(n) = \{n\times n \text{ cx. matrices} \,|\, A^* = -A\}$ of skew-hermitian matrices.

Alternatively, one can show $T_I U(n) \subset \mathfrak{u}(n)$ and then use a dimension count. For this, note that if $A = B + Ci$ then $A^* = A$ iff B is symmetric and C is skew-symmetric. Hence

$$\dim H(n) = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2 \quad \text{and similarly} \qquad \dim \mathfrak{u}(n) = n^2.$$

4. (a) For any $x, g, h \in G$ we have $L_g L_h(x) = L_g(hx) = ghx = L_{gh}(x)$. As special cases we get $L_g L_{g^{-1}}(x) = x$ and $L_{g^{-1}} L_g(x) = x$. Thus $L_g L_h = L_{gh}$ and L_g is a diffeomorphism with inverse $L_{g^{-1}}$.

   (b) By assumption there is a neighborhood U of $I \in G$ and a chart $\phi : U \to V \subset \mathbf{R}^n$ on which the group operations are smooth. For each $g \in G$ set $U_g = L_g(U)$ and let $\phi_g : U_g \to V$ by $\phi_g = \phi \circ L_{g^{-1}}$. Then U_g is a neighborhood of g and ϕ_g is a bijection because it is the composition of two bijection). Define an atlas by

$$\mathcal{A} \;=\; \{(U_g, \phi_g) \,|\, g \in G\}.$$

   These U_g cover G. We will show that whenever $U_g \cap U_h \neq \emptyset$ the transition map $\phi_h^{-1}\phi_g : U_g \cap U_h \to U_g \cap U_h$ is smooth. For this, first note that

$$\phi_h \phi_g^{-1} \;=\; \phi \circ L_{h^{-1}} \,(\phi \circ L_{g^{-1}})^{-1} \;=\; \phi \circ L_{h^{-1}} \circ L_g \circ \phi^{-1} \;=\; \phi \circ L_{h^{-1}g} \circ \phi^{-1}.$$

   This looks smooth, but be careful: *we do not know that L_g is smooth or even continuous* for $g \notin U$. To deal with this problem, fix $x \in U_g \cap U_h$. Since $x \in U_h = L_h U$ we have $h^{-1}x \in U$, so $L_{h^{-1}x}$ is smooth by the hypothesis. Similarly, since $x \in U_g$ we have $g^{-1}x \in U$, so $L_{g^{-1}x}$ is smooth and hence so is its inverse. Therefore

$$L_{h^{-1}g} = L_{(h^{-1}x)(x^{-1}g)} = L_{h^{-1}x} \circ \left(L_{g^{-1}x}\right)^{-1}$$

   is smooth. This shows that all transition maps are smooth, so $\mathcal{A}$ defines a differentiable structure on G.

   (c) The exponential map $A \mapsto e^A$ for skew-hermitian A is a map $\exp : \mathfrak{u}(n) \to U(n)$ with $\exp(0) = I$ and $d\exp_0 = Id$. By the Inverse Function Theorem it is a local diffeomorphism at the identity. This means that there is a neighborhood U of $I \in U(n)$ and a chart $\exp^{-1} : U \to V \subset \mathbf{R}^m$ where $m = \dim U(n)$. Therefore $U(n)$ is a manifold by part (b).

5. If $g = ai + bj + ck$ is a pure imaginary unit quaternion then $g\bar{g} = 1$, so $g^{-1} = \bar{g} = -g$. Then

$$\begin{aligned}
gig^{-1} = (ai+bj+ck)i(-ai-bj-ck) &= (ai+bj+ck)(a-bk+cj) \\
&= (a^2-b^2-c^2)i + 2abj + 2ack \\
gjg^{-1} = (ai+bj+ck)j(-ai-bj-ck) &= (ai+bj+ck)(ak+b-ci) \\
&= 2abi + (-a^2+b^2-c^2)j + 2bck \\
gkg^{-1} = (ai+bj+ck)k(-ai-bj-ck) &= (ai+bj+ck)(-aj+bi+c) \\
&= 2aci + 2bcj + (-a^2-b^2+c^2)i.
\end{aligned}$$

Hence

$$\mathrm{Ad}_g \;=\; \begin{pmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & -a^2+b^2-c^2 & 2bc \\ 2ac & 2bc & -a^2-b^2+c^2 \end{pmatrix}$$