Math 868 — Homework 2

Due Monday, Sept. 17

1. Let $R = \{(x,y)\,|\,x>0\}$ be the right half-plane and let $\Phi : \mathbf{R}^+ \times (-\pi/2, \pi/2) \to R$ be the map $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$ that changes into polar coordinates into xy coordinates. Write down $D\Phi$ and $D\Phi^{-1}$ as matrices.

2. (a) Suppose that $f : M \to N$ is a diffeomorphism between manifolds. Prove that at each point p, Df_p is an isomorphism of the tangent spaces.
   (b) Using (a), prove that $\mathbf{R}^k$ is not diffeomorphic to $\mathbf{R}^n$ if $k \neq n$.
   (c) If a nonempty smooth n-manifold is diffeomorphic to an m-manifold, prove that $n = m$ (This is Problem 3-3 on page 79 in Lee's textbook)

3. (Lee, Problem 3-1 page 78) Suppose that $f : M \to N$ is a smooth map between manifolds with M connected and that $Df_p : T_pM \to T_{f(p)}N$ is the zero map at each $p \in M$. Prove that the image of f is a single point. *Hint:* first, for each $p \in M$, use charts around p and $q = f(p)$ to show that f is locally constant.

4. Suppose that $f : M \to N$ is a smooth map between manifolds, $\phi : U \to V \subset M$ and $\psi : \hat{U} \to \hat{V} \subset N$ are charts with $U \subset \mathbf{R}^m$, $\hat{U} \subset \mathbf{R}^m$, and $f(V) \subset \hat{V}$. As in class, define
$$Df_p \;:=\; D\psi \circ Dh \circ D\phi^{-1} : T_pM \to T_qN$$
where $q = f(p)$. Prove that this definition is independent of the charts ϕ and ψ.

5. In class we proved that T_pM is the set of velocity vectors at p for all paths in M through p. Use this to write down two vectors in $\mathbf{R}^3$ that span T_pS^2 at the point $p = (a,b,c)$ (this gives an second solution to Problem 7b on HW Set 1)

6. Use the matrices you found in Problem 1 above to answer Problem 4.5 on page 101 of Lee.

7. Fix a point p in a manifold M. Let I_p denote the set of all $f \in C^\infty(M)$ with $f(p) = 0$ and let $I_p^2 =$ ideal generated by $\{f^2\,|\,f \in I_p\}$.

   (a) Show that I_p is an ideal in the ring $C^\infty(M)$ (look up the definition of 'ideal' in an algebra book if necessary) and that I_p^2 is a vector subspace of I_p.
   (b) Using the definition of derivation, show that any derivation $X : C^\infty(M) \to \mathbf{R}$ at p satisfies $Xf = 0$ for all $f \in I_p^2$.
   (c) Show that any quadratic function $f = ax^2 + bxy + cy^2$ on $\mathbf{R}^2$ lies in I_0^2 (complete the square!)
   (d) A function f on $\mathbf{R}^n$ *vanishes to second order at the origin* if there is a constant c such that $|f| \le cr^2$ in some ball $B(0,\epsilon)$ around the origin. Prove that
$$I_0^2 \;=\; \{f \in C^\infty(\mathbf{R}^n) \mid f \text{ vanishes to second order at the origin}\}$$
   Hint: As in class, it follows from the Fundamental Theorem of Calculus that we can write $f(x^1, \ldots x^n) = f(0) + \sum x^i a_i + \sum_{i,j} x^{ij} b_{ij}(x^1, \ldots x^n)$ *for some smooth functions* b_{ij}.

   Conclusion: Clearly, part (d) implies that $I_p^2 = \{f \in C^\infty(M)\,|\,f \text{ that vanish to second order at } p\}$. Thus (b) and (d) show that any derivation $X : C^\infty(M) \to \mathbf{R}$ at p satisfies $Xf = 0$ for every functions f that vanishes to second order at p.

8. Consider a smooth function $f : M \to \mathbf{R}$ as a map between manifolds. Then $df_p : T_pM \to T_{f(p)}\mathbf{R}$.

   (a) Show there is a natural identification $T_t\mathbf{R} = \mathbf{R}$, so we can consider df_p as an element of the dual space T_p^*M.
   (b) Show that in coordinates $df_p = \sum \left(\frac{\partial f}{\partial x^i}\right)_p dx^i$ where $\{dx^i\}$ are the dual basis of $\{\frac{\partial f}{\partial x^i}\}$ of T_pM.