# Math 868 — Homework 12

**Due Friday, Dec 7**

These problems relate to pages 273 – 284 in Lee's textbook.

1. Do Lee's Problem 11-13: Let $(M, g)$ and $\tilde{M}, \tilde{g})$ be Riemannian manifolds. Suppose $f : M \to \tilde{M}$ is a smooth map such that $f^*\tilde{g} = g$. Show that $f$ is an immersion.

2. Do Lee's Problem 11-16: Show that the shortest distance between two points in Euclidean space is a straight segment. More precisely, for $p, q \in \mathbf{R}^n$, let $\gamma_0 : [0, 1] \to \mathbf{R}^n$ be the path

$$\gamma_0(t) = (1 - t)p + tq.$$

and let $\gamma : [0, 1] \to \mathbf{R}^n$ be another path with $\gamma(0) = p$ and $\gamma(1) = q$. Prove that

$$L_{g_0}(\gamma_0) \leq L_{g_0}(\gamma).$$

Hint: choose coordinates with origin at $p$ and with $q = (1, 0, \dots, 0)$, write $\gamma(t) = \gamma_0(t) + x(t)$ and note that $|\dot{\gamma}_0 + \dot{x}| \geq \dot{\gamma}_0 + \dot{x}^1$.

3. Let $(M, g)$ be a Riemannian manifold, $f \in C^\infty(M)$ and let $\nabla f$ be the gradient of $f$, defined by $g(\nabla f, X) = df(X)$ for all vectors $X$.

   (a) Prove that if $q$ is a regular value of $f$ then at each point $p$ in the level set $M_q = f^{-1}(q)$, $(\nabla f)_p$ is non-zero and perpendicular to the tangent space to the level set $M_q$.

   (b) Use orientation forms to show that if $M$ is orientable then so is the level set at each regular value $q$.

4. Let $(M, g)$ be an oriented Riemannian manifold. As in class, the metric determines a volume form that is given in each positively-oriented coordinate chart by

$$dvol_g = \sqrt{\det g_{ij}}\; dx^1 \wedge \cdots \wedge dx^n.$$

Show directly that this does not depend on the choice of the positively-oriented chart, that is, if $\phi(y^1, \dots y^n) = (x^1, \dots, x^n)$ is a diffeomorphism between charts then $\phi^* dvol_g$ has the same expression in $y$-coordinates. (Use the properties of determinants and the formula related determinants to wedge products).