Math 868 — Homework 11

Due Wednesday, Dec 5

These problems relate to Chapter 15 in Lee's textbook. *Reading Chapter 15 is recommended!.*

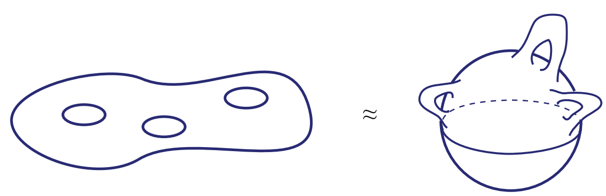
1. Show that if a manifold M is the disjoint union of two components U and V, then its DeRham cohomology is $H^*(M) = H^*(U) \oplus H^*(V)$.

2. Read the statement and proof of the "Zigzag Lemma" on page 395-6 of Lee (also done in class). Finish the proof be verifying the statements listed in the last paragraph of the proof, namely:
   (a) The cohomology class $[a]$ is independent of the choices made along the way (this was partially done in class),
   (b) δ is linear, and
   (c) The resulting long exact sequence is exact.

3. Compute the DeRham cohomology groups of the torus T^2 as follows.
   (a) Explain, in a few words, why the cylinder $C = (0,3) \times S^1$ and the circle have the same DeRham cohomology groups $H^p(C) = H^p(S^1)$ for $p = 0, 1, 2$.
   (b) T^2 can be regarded as the union of two cylinders $C_1 = (0,3) \times S^1$ and $C_2 = (1,4) \times S^1$ whose intersection is the disjoint union of open sets $U = (1,2) \times S^1$ and $V = (3,4) \times S^1$, which are both cylinders. *Draw a picture of this.*
   (c) Compute the DeRham cohomology groups $H^p(T^2)$ for $p = 0, 1, 2$.

4. (Lee Problem 15-1) Compute the DeRham cohomology groups of $\mathbf{R}^n$ minus two points (Use Corollary 15.19 on page 403 of Lee).

5. Let Σ_g be a compact, orientable 2-manifold of genus g; Σ_g is diffeomorphic to a sphere S^2 with g handles attached.



   Use induction (starting from the $g = 0$ case) and Mayer-Vietoris to prove that $\begin{cases} H^0(M) = \mathbf{R} \\ H^1(M) = \mathbf{R}^{2g} \\ H^2(M) = \mathbf{R}. \end{cases}$

6. Let (M, ω) be a compact $2n$-dimensional symplectic manifold (without boundary). In Problem 5 of Homework Set 9 you showed that the n-fold wedge product $\omega^k = \omega \wedge \cdots \wedge \omega$ is a nowhere zero $2n$-form. Thus every symplectic manifold is orientable, and ω^n specified an orientation. Use this fact to answer Lee's Problem 15-8, namely:
   (a) Show that ω^n is not exact (Use Stokes' Theorem)
   (b) Show that $H^{2k}(M) \neq 0$ for $k = 1, \ldots n$ because $[\omega^k] \neq 0$.
   (c) Show that the only sphere that admits a symplectic structure is S^2.