Math 868 — Homework 1

Due Weds, Sept. 5

1. Prove that a bijective map between manifolds need not be a diffeomorphism. In fact, show that $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^3$ is an example.

2. The *graph* of a function $f : M \to \mathbf{R}$ on a manifold M is the set

$$G_f \;=\; \{(x, f(x)) \,|\, x \in M\}.$$

   Define $\phi : M \to G_f$ by $\phi(x) = (x, f(x))$. Prove that if f is smooth then ϕ is a diffeomorphism; this means that the graph is a manifold.

3. Let T be the torus consisting of all points in $\mathbf{R}^3$ that are distance 1 from the circle of radius 2 in the xy-plane. Prove that T is diffeomorphic to $S^1 \times S^1$ by writing down a map $\phi : S^1 \times S^1 \to T$ and showing that ϕ is 1-1, onto, smooth and has a smooth inverse.

4. Prove that one cannot parametrize the sphere S^n by a single chart $\phi : S^n \to V \subset \mathbf{R}^n$. (*Hint:* the sphere is compact.)

5. Do Problem 1-5 on page 28 of the textbook (on stereographic projection).

6. This problem gives the steps for constructing a "C^∞ bump function". Pages 49-51 in the textbook describe a similar — but not identical — construction.

   (a) An extremely useful function $f : \mathbf{R} \to \mathbf{R}$ is

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

   Sketch the graph of f and prove that f is smooth (use the fact that the composition of smooth functions is smooth, that e^x and $1/x^2$ (for $x \neq 0$) are smooth, and explicitly show that all derivatives of f are continuous at $x = 0$).

   (b) Fix $0 < a < b$. Sketch the graph of $g(x) = f(x - a)f(b - x)$ and show that g is a smooth function, positive on the interval (a, b) and 0 elsewhere.

   (c) Sketch the graph of

$$h(x) \;=\; \frac{\int_{-\infty}^{x} g \; dx}{\int_{-\infty}^{\infty} g \; dx}$$

   This is a smooth function satisfying $h(x) = 0$ for $x < a$, $h(x) = 1$ for $x > b$ and $0 < h(x) < 1$ for all $x \in (a, b)$ (no proof needed here).

   (d) Now construct a smooth "bump function" $\beta(x)$ on $\mathbf{R}^n$ that equals 1 on the ball $B(0, a)$, is zero outside the ball $B(0, b)$ and is strictly between 0 and 1 at the intermediate points.

7. (a) The tangent space to the unit circle S^1 at $p = (a, b)$ is a 1-dimensional subspace of R^2. Use the definition of tangent space to prove that T_pS^1 is the span of $(-b, a)$.

   (b) Similarly use the definition of tangent space to find two vectors that are a basis of T_pS^2 at a point $p = (a, b, c)$. (This can be done by simple geometry *but don't do that.* Use the definition of tangent space).