# Chapter 2

# Vector Spaces and Bases

This chapter and the next are an overview of elementary linear algebra. The reader is expected to be already familiar with basic matrix operations such as matrix multiplication, row reduction, taking determinants and finding inverses. However, no prior knowledge of vector spaces, bases, dimension, or linear transformations is assumed. That material is covered quickly from the beginning.

## 2.1 Vector Spaces

Linear algebra is the study of vector spaces. A vector space is a set whose elements (called *vectors*, naturally) can be added to each other and multiplied by scalars, such that the usual rules (such as "$\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$") apply. By *scalars* we usually mean real numbers, in which case we say we are dealing with a *real vector space*. Sometimes by scalars we mean complex numbers, in which case we are dealing with a *complex vector space*.

In principle, we could consider scalars that belong to an arbitrary field, not just to $\mathbb{R}$ or $\mathbb{C}$. Algebraic geometers, number theorists and computer scientists often consider vector spaces over finite fields or over the rational numbers. In this book, however, we will only consider real and complex vector spaces.

For completeness, the axioms for a vector space are listed on page 10. Instead of dwelling on the axioms, however, consider some examples:

1. The set of all real $n$-tuples $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, together with the

operations

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}; \quad c\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix}, \quad (2.1)$$

is a real vector space denoted $\mathbb{R}^n$. Since columns are difficult to type, we will usually write $\mathbf{x} = (x_1, \ldots, x_n)^T$. The superscript "T", which stands for "transpose", indicates that we are thinking of the $n$ numbers forming a column rather than a row. (The reason for making $\mathbf{x}$ a column rather than a row is so we can take the product $A\mathbf{x}$, where $A$ is an $m \times n$ matrix.)

2. If in example 1 we allow the entries $x_i$ and the scalars $c$ to be complex numbers, then we have a complex vector space denoted $\mathbb{C}^n$.

3. Let $M_{nm}$ be the space of real $n \times m$ matrices. Matrices can be added and multiplied by real numbers, so $M_{nm}$ is a real vector space.

4. Let $C^0[0,1]$ be the set of continuous real-valued functions on the closed interval $[0,1]$. Continuous functions may be added and multiplied by real numbers, yielding other continuous functions, so $C^0[0,1]$ is a real vector space.

In examples 3 and 4, we considered real-valued matrices and functions to obtain real vector spaces. If we instead consider complex-valued matrices or functions, we obtain a complex vector space.

With these examples under our belts, we now examine the formal definition:

**Definition** *Let $V$ be a set on which addition and scalar multiplication are defined. (That is, if $\mathbf{x}$ and $\mathbf{y}$ are elements of $V$, and $c$ is a scalar, then $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$ are elements of $V$.) If the following eight axioms are satisfied by all elements $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ of $V$ and all scalars $a$ and $b$, then $V$ is called a* vector space *and the elements of $V$ are called* vectors. *If these axioms apply to multiplication by real scalars, then $V$ is called a* real vector space. *If the axioms apply to multiplication by complex scalars, then $V$ is a* complex vector space.

1. *Commutativity of addition:* $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$.

3. *Additive identity: There exists a vector, denoted* $\mathbf{0}$, *such that, for every vector* $\mathbf{x}$, $\mathbf{0}+\mathbf{x}=\mathbf{x}$.

4. *Additive inverses: For every vector* $\mathbf{x}$ *there exists a vector* $(-\mathbf{x})$ *such that* $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$.

5. *First distributive law:* $a(\mathbf{x}+\mathbf{y})=a\mathbf{x}+a\mathbf{y}$.

6. *Second distributive law:* $(a+b)\mathbf{x}=a\mathbf{x}+b\mathbf{x}$.

7. *Multiplicative identity:* $1(\mathbf{x})=\mathbf{x}$.

8. *Relation to ordinary multiplication:* $(ab)\mathbf{x}=a(b\mathbf{x})$.

In practice, checking these axioms is a tedious and often pointless task. Addition and scalar multiplication are usually defined in a straightforward way that makes these axioms obvious. What is far *less* obvious is that addition and scalar multiplication make sense as operations *on* $V$. One must check that the sum of two arbitrary elements of $V$ is in $V$ and that the product of an arbitrary scalar and an arbitrary element of $V$ is in $V$.

**Definition** *A set* $S$ *is* closed under addition *if the sum of any two elements of* $S$ *is in* $S$, *and is* closed under scalar multiplication *if the product of an arbitrary scalar and an arbitrary element of* $S$ *is in* $S$.

Frequently we consider a subset $W$ of a vector space $V$. In this case, addition and scalar multiplication are already defined, and already satisfy the eight axioms. If $W$ is closed under addition and scalar multiplication, then $W$ is a vector space in its own right, and we call $W$ a *subspace* of $V$.

With this in mind we consider a few more examples of vector spaces, as well as some sets that are not vector spaces. See Figure 2.1.

1. Let $W=\{\mathbf{x}\in\mathbb{R}^2 | x_1+x_2=0\}$. The sum of any two vectors in $W$ is in $W$, and any scalar multiple of a vector in $W$ is in $W$. (Check this!) $W$ is a subspace of the vector space $\mathbb{R}^2$.

2. More generally, let $A$ be any fixed $m\times n$ matrix ...

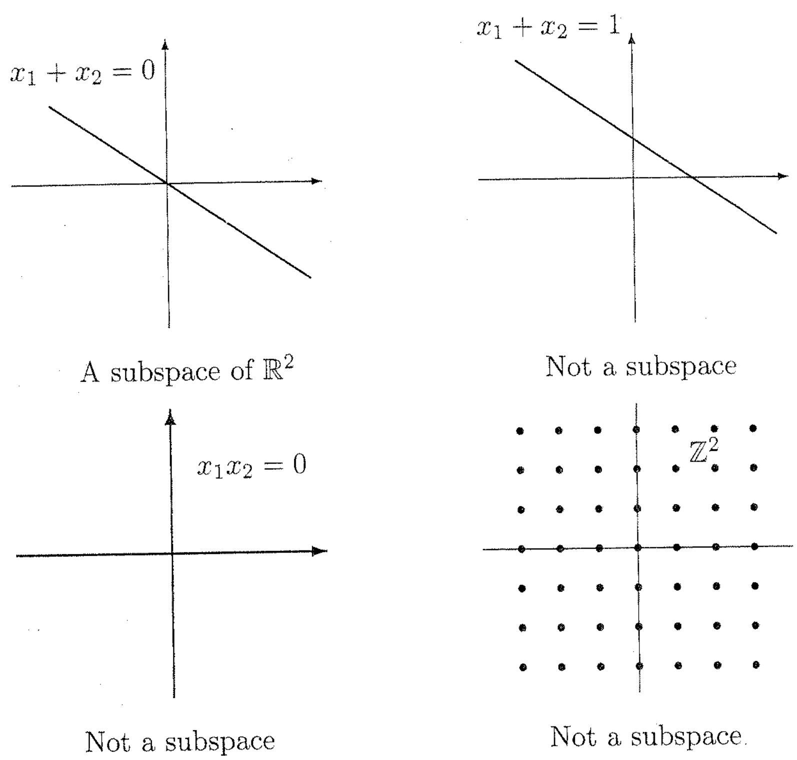


**Figure 2.1:** Four subsets of $\mathbb{R}^2$

3. Let $\mathbb{R}[t]$ be the set of all real-valued polynomials in a fixed variable $t$. Polynomials are continuous functions on $[0, 1]$, so $\mathbb{R}[t]$ is a subset of $C^0[0, 1]$. The sum of two polynomials is a polynomial, as is the product of scalar and a polynomial, so $\mathbb{R}[t]$ is a subspace of $C^0[0, 1]$.

4. Let $\mathbb{R}_n[t]$ be the set of real polynomials of degree $n$ or less. For $n < m$, $\mathbb{R}_n[t]$ is a subspace of $\mathbb{R}_m[t]$, and $\mathbb{R}_n[t]$ is always a subspace of $\mathbb{R}[t]$.

5. Instead of considering polynomials with real coefficients, we could consider polynomials with complex coefficients to get examples of complex vector spaces. The space of polynomials with complex coefficients is usually denoted $\mathbb{C}[t]$.

6. Let $W' = \{\mathbf{x} \in \mathbb{R}^2 | x_1 + x_2 = 1\}$. $W'$ is not a vector space, as the sum of two elements of $W'$, or a scalar multiple of an element of $W'$, is typically not in $W'$. (Again, check this!)

7. Let $\mathbb{Z}^2 = \{\mathbf{x} \in \mathbb{R}^2 | x_1$ and $x_2$ are integers $\}$. $\mathbb{Z}^2$ is closed under addition, but not under scalar multiplication. $\mathbb{Z}^2$ is a sub*set* of $\mathbb{R}^2$, but not a sub*space*.

8. The set $\{\mathbf{x} \in \mathbb{R}^2 | x_1 x_2 = 0\}$ is the union of the coordinate axes in $\mathbb{R}^2$. This set is closed under scalar multiplication but not under addition, and so is not a subspace of $\mathbb{R}^2$.

Finally, notice that length is not an essential property of a vector. What is the length of a polynomial, or of a continuous function? It is true that many vector spaces come equipped with a notion of length, and this concept can be extremely useful. However, many vector spaces do not come so equipped. In Chapters 2–5 we concentrate on those operations that make sense in all vector spaces. In Chapters 6 and beyond we examine what more can be done when a vector space has an inner product and a notion of length.

## Exercises

Explain why each of the following sets is or is not a vector space.

**1.** $\{\mathbf{x} \in \mathbb{R}^2 | x_1 \leq x_2\}$.

**2.** $\{\mathbf{x} \in \mathbb{R}^2 | x_1 + x_2 = 0$ and $x_1$ is rational. $\}$

**3.** $\{\mathbf{x} \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 = 1\}$.

**4.** $\{\mathbf{x} \in \mathbb{R}^n | x_1^2 \geq x_2^2 + \cdots + x_n^2\}$.

**5.** All vectors $\mathbf{x} \in \mathbb{R}^4$ that satisfy $x_1 + x_2 = x_3 - x_4$ and $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$.

**6.** All vectors in $\mathbb{R}^3$ of the form $c_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix}$, where $c_1$ and $c_2$ are arbitrary real numbers.

**7.** All vectors in a vector space $V$ of the form $c_1\mathbf{v} + c_2\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are fixed vectors in $V$ and $c_1$ and $c_2$ are arbitrary scalars.

**8.** All polynomials $\mathbf{p}(t) \in \mathbb{R}[t]$ such that $\mathbf{p}(3) = 0$.

**9.** All polynomials $\mathbf{p}(t) \in \mathbb{R}[t]$ such that $\mathbf{p}(3) = 1$.

**10.** All non-negative functions in $C^0[0, 1]$.

**11.** All polynomials in $\mathbb{R}_5[t]$ with integer coefficients.

**12.** Let $X$ be an arbitrary set. Is the set of all integer-valued func-