17. Apply the simplex method to the problem of finding a non-negative solution of

$$\begin{aligned} 6x_1 + 3x_2 - 4x_3 - 9x_4 - 7x_5 - 5x_6 &= 0 \\ -5x_1 - 8x_2 + 8x_3 + 2x_4 - 2x_5 + 5x_6 &= 0 \\ -2x_1 + 6x_2 - 5x_3 + 8x_4 + 8x_5 + x_6 &= 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= 1. \end{aligned}$$

This is equivalent to Exercise 5 of Section 1.

4 | Applications to Communication Theory

This section requires no more linear algebra than the concepts of a basis and the change of basis. The material in the first four sections of Chapter I and the first four sections of Chapter II is sufficient. It is also necessary that the reader be familiar with the formal elementary properties of Fourier series.

Communication theory is concerned largely with signals which are uncertain, uncertain to be transmitted and uncertain to be received. Therefore, a large part of the theory is based on probability theory. However, there are some important concepts in the theory which are purely of a vector space nature. One is the sampling theorem, which says that in a certain class of signals a particular signal is completely determined by its values (samples) at an equally spaced set of times extending forever.

Although it is usually not stated explicitly, the set of functions considered as signals form a vector space over the real numbers; that is, if $f(t)$ and $g(t)$ are signals, then $(f + g)(t) = f(t) + g(t)$ is a signal and $(af)(t) = af(t)$, where a is a real number, is also a signal. Usually the vector space of signals is infinite dimensional so that while many of the concepts and theorems developed in this book apply, there are also many that do not. In many cases the appropriate tool is the theory of Fourier integrals. In order to bring the topic within the context of this book, we assume that the signals persist for only a finite interval of time and that there is a bound for the highest frequency that will be encountered. If the time interval is of length 1, this assumption has the implication that each signal $f(t)$ can be represented as a finite series of the form

$$f(t) = \tfrac{1}{2}a_0 + \sum_{k=1}^{N} a_k \cos 2\pi kt + \sum_{k=1}^{N} b_k \sin 2\pi kt. \tag{4.1}$$

Formula (4.1) is in fact just a precise formulation of the vague statement that the highest frequency to be encountered is bounded. Since the coefficients can be taken to be arbitrary real numbers, the set of signals under consideration forms a real vector space V of dimension $2N + 1$. We show that $f(t)$ is determined by its values at $2N + 1$ points equally spaced in time. This statement is known as the *finite sampling theorem*.

The classical infinite sampling theorem from communication theory requires an assumption analogous to the assumption that the highest frequencies are bounded. Only the assumption that the signal persists for a finite interval of time is relaxed. In any practical problem some bound can be placed on the duration of the family of signals under consideration. Thus, the restriction on the length of the time interval does not alter the significance or spirit of the theorem in any way.

Consider the function

$$\psi(t) = \frac{1}{2N+1}\left(1 + 2\sum_{k=1}^{N} \cos 2\pi kt\right) \in V. \tag{4.2}$$

$$\psi(t) = \frac{\sin \pi t + \sum_{k=1}^{N} 2\cos 2\pi kt \sin \pi t}{(2N+1)\sin \pi t}$$

$$= \frac{\sin \pi t + \sum_{k=1}^{N} \sin(2\pi kt + \pi t) - \sin(2\pi kt - \pi t)}{(2N+1)\sin \pi t}$$

$$= \frac{\sin(2N+1)\pi t}{(2N+1)\sin \pi t} \tag{4.3}$$

From (4.2) we see that $\psi(0) = 1$, and from (4.3) we see that $\psi(j/2N+1) = 0$ for $0 < |j| \le N$.

Consider the functions

$$\psi_k(t) = \psi\left(t - \frac{k}{2N+1}\right), \qquad \text{for } k = -N, -N+1, \ldots, N. \tag{4.4}$$

These $2N + 1$ functions are all members of V. Furthermore, for $t_j = j/(2N+1)$ we see that $\psi_j(t_j) = 1$ while $\psi_k(t_j) = 0$ for $k \neq j$. Thus, the $2N + 1$ functions obtained are linearly independent. Since V is of dimension $2N + 1$, it follows that the set $\{\psi_k(t) \mid k = -N, \ldots, N\}$ is a basis of V. These functions are called the *sampling functions*.

If $f(t)$ is any element of V it can be written in the form

$$f(t) = \sum_{k=-N}^{N} d_k \psi_k(t). \tag{4.5}$$

However,

$$f(t_j) = \sum_{k=-N}^{N} d_k \psi_k(t_j) = d_j, \tag{4.6}$$

or

$$f(t) = \sum_{k=-N}^{N} f(t_k)\psi_k(t). \tag{4.7}$$

Thus, the coordinates of $f(t)$ with respect to the basis $\{\psi_k(t)\}$ are $(f(t_{-N}), \ldots, f(t_N))$, and we see that these samples are sufficient to determine $f(t)$.

It is of some interest to express the elements of the basis $\{\frac{1}{2}, \cos 2\pi t, \ldots, \sin 2\pi Nt\}$ in terms of the basis $\{\psi_k(t)\}$.

$$\begin{aligned} \frac{1}{2} &= \sum_{k=-N}^{N} \tfrac{1}{2}\psi_k(t) \\ \cos 2\pi jt &= \sum_{k=-N}^{N} \cos 2\pi jt_k \psi_k(t) \\ \sin 2\pi jt &= \sum_{k=-N}^{N} \sin 2\pi jt_k \psi_k(t). \end{aligned} \tag{4.8}$$

Expressing the elements of the basis $\{\psi_k(t)\}$ in terms of the basis $\{\frac{1}{2}, \cos 2\pi t, \ldots, \sin 2\pi Nt\}$ is but a matter of the definition of the $\psi_k(t)$:

$$\begin{aligned} \psi_k(t) &= \psi\left(t - \frac{k}{2N+1}\right) \\ &= \frac{1}{2N+1}\left[1 + 2\sum_{j=1}^{N} \cos 2\pi j\left(t - \frac{k}{2N+1}\right)\right] \\ &= \frac{1}{2N+1}\left(1 + 2\sum_{j=1}^{N} \cos \frac{2\pi jk}{2N+1} \cos 2\pi jt + 2\sum_{j=1}^{N} \sin \frac{2\pi jk}{2N+1} \sin 2\pi jt\right). \end{aligned} \tag{4.9}$$

With this interpretation, formula (4.1) is a representation of $f(t)$ in one coordinate system and (4.7) is a representation of $f(t)$ in another. To express the coefficients in (4.1) in terms of the coefficients in (4.7) is but a change of coordinates. Thus, we have

$$\begin{aligned} a_j &= \frac{2}{2N+1} \sum_{k=-N}^{N} f(t_k) \cos \frac{2\pi jk}{2N+1} = \frac{2}{2N+1} \sum_{k=-N}^{N} f(t_k) \cos 2\pi jt_k \\ b_j &= \frac{2}{2N+1} \sum_{k=-N}^{N} f(t_k) \sin \frac{2\pi jk}{2N+1} = \frac{2}{2N+1} \sum_{k=-N}^{N} f(t_k) \sin 2\pi jt_k. \end{aligned} \tag{4.10}$$

There are several ways to look at formulas (4.10). Those familiar with the theory of Fourier series will see the a_j and b_j as Fourier coefficients with formulas (4.10) using finite sums instead of integrals. Those familiar with probability theory will see the a_j as covariance coefficients between the samples of $f(t)$ and the samples of $\cos 2\pi jt$ at times t_k. And we have just viewed them as formulas for a change of coordinates.

If the time interval had been of length T instead of 1, the series corresponding to (4.1) would be of the form

$$f(t) = \tfrac{1}{2}a_0 + \sum_{k=1}^{N} a_k \cos \frac{2\pi k}{T} t + \sum_{k=1}^{N} b_k \sin \frac{2\pi k}{T} t. \tag{4.11}$$

The vector space would still be of dimension $2N + 1$ and we would need $2N + 1$ samples spread equally over an interval of length T, or $(2N + 1)/T$ samples per unit time. Since $N/T = W$ is the highest frequency present in the series (4.11), we see that for large intervals of time approximately $2W$ samples per unit time are required to determine the signal. The infinite sampling theorem, referred to at the beginning of this section, says that if W is the highest frequency present, then $2W$ samples per second suffice to determine the signal. The spirit of the finite sampling theorem is in keeping with the spirit of the infinite sampling theorem, but the finite sampling theorem has the practical advantage of providing effective formulas for determining the function $f(t)$ and the Fourier coefficients from the samples.

BIBLIOGRAPHY NOTES

For a statement and proof of the infinite sampling theorem see P. M. Woodward, *Probability and Information Theory, with Applications to Radar.*

EXERCISES

1. Show that $\psi(t_r - t_s) = \psi\left(\dfrac{r-s}{2N+1}\right) = \delta_{rs}$ for $-N \le r, s \le N$.

2. Show that if $f(t)$ can be represented in the form of (4.1), then

$$a_k = 2\int_{-1/2}^{1/2} f(t)\cos 2\pi kt\,dt, \qquad k = 0, 1, \dots, N,$$

$$b_k = 2\int_{-1/2}^{1/2} f(t)\sin 2\pi kt\,dt, \qquad k = 1, \dots, N.$$

3. Show that $\displaystyle\int_{-1/2}^{1/2} \psi(t)\,dt = \frac{1}{2N+1}$.

4. Show that $\displaystyle\int_{-1/2}^{1/2} \psi_k(t)\,dt = \frac{1}{2N+1}$.

5. Show that if $f(t)$ can be represented in the form (4.7), then

$$\int_{-1/2}^{1/2} f(t)\,dt = \sum_{k=-N}^{N} \frac{1}{2N+1} f(t_k).$$

This is a formula for expressing an integral as a finite sum. Such a formula i called a *mechanical quadrature*. Such formulas are characteristic of the theory o orthogonal functions.

6. Show that

$$\sum_{k=-N}^{N} \frac{1}{2N+1} \cos 2\pi r t_k = \delta_{rk},$$

and

$$\sum_{k=-N}^{N} \frac{1}{2N+1} \sin 2\pi r t_k = 0$$

7. Show that

$$\sum_{k=-N}^{N} \frac{1}{2N+1} \cos 2\pi r(t - t_k) = \delta_{rk}$$

and

$$\sum_{k=-N}^{N} \frac{1}{2N+1} \sin 2\pi r(t - t_k) = 0.$$

8. Show that

$$\sum_{k=-N}^{N} \psi_k(t) = 1.$$

9. Show that

$$f(t) = \sum_{k=-N}^{N} f(t)\psi_k(t).$$

10. If $f(t)$ and $g(t)$ are integrable over the interval $[-\frac{1}{2}, \frac{1}{2}]$, let

$$(f, g) = 2\int_{-1/2}^{1/2} f(t)g(t)\,dt.$$

Show that if $f(t)$ and $g(t)$ are elements of V, then (f, g) defines an inner product in V. Show that $\left\{\frac{1}{\sqrt{2}}, \cos 2\pi t, \ldots, \sin 2\pi N t\right\}$ is an orthonormal set.

11. Show that if $f(t)$ can be represented in the form (4.1), then

$$2\int_{-1/2}^{1/2} f(t)^2\,dt = \frac{a_0^2}{2} + \sum_{k=1}^{N} a_k^2 + \sum_{k=1}^{N} b_k^2.$$

Show that this is Parseval's equation of Chapter V.

12. Show that

$$\int_{-1/2}^{1/2} \cos 2\pi r t\ \psi(t)\,dt = \frac{1}{2N+1}.$$

13. Show that

$$\int_{-1/2}^{1/2} \cos 2\pi r t\ \psi_k(t)\,dt = \frac{1}{2N+1} \cos 2\pi r t_k.$$

14. Show that

$$\int_{-1/2}^{1/2} \sin 2\pi r t\ \psi_k(t)\,dt = \frac{1}{2N+1} \sin 2\pi r t_k.$$

15. Show that

$$\int_{-1/2}^{1/2} \psi_r(t)\psi_s(t)\,dt = \frac{1}{2N+1}\,\delta_{rs}.$$

16. Using the inner product defined in Exercise 10 show that $\{\psi_k(t) \mid k = -N, \ldots, N\}$ is an orthonormal set.