Math 415

Homework Set 8

due Friday, March 23

Do the following problems. You may use a calculator to find the square roots and cosines. Minimize the number of decimal places.

1. Find the length of $\mathbf{u} = (7, 11)$, $\mathbf{v} = (2, 3, 4)$ and $\mathbf{w} = (2, 3, 4, 5)$.
2. Find the angle between (a) $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (2, 3, 4)$, (b) $\mathbf{u} = (1, -1, 2, -2)$ adn $\mathbf{v} = (2, 3, 4, 5)$.
3. For each pair of vectors, determnie whether the angle between them is acute ($< 90°$), obtuse ($> 90°$) or right.
   (a) $\mathbf{u} = (2, -2)$ and $\mathbf{v} = (5, 4)$.
   (b) $\mathbf{u} = (2, 3, 4)$ and $\mathbf{v} = (2, -8, 5)$.
   (c) $\mathbf{u} = (1, -1, 1, -1)$ and $\mathbf{v} = (3, 4, 5, 3)$.
4. For which choice of k are the vectors $\mathbf{u} = (2, 3, 4)$ and $\mathbf{v} = (1, k, 1)$ orthogonal?
5. Consider the vectors $\mathbf{u} = (1, 1, 1, \cdots, 1)$ and $\mathbf{v} = (1, 0, 0, 0, \cdots, 0)$ in $\mathbb{R}^n$. Determine the angle between then for $n = 2, 3, 4$ and find the limit of this angle as $n \to \infty$.
6. Find the orthogonal projection of $\mathbf{u} = (49, 49, 49)$
   (a) onto the vector $\mathbf{v} = (2, 3, 6)$
   (b) onto the subspace spanned by $\mathbf{v} = (2, 3, 6)$ and $\mathbf{w} = (3, -6, 2)$.
7. Find the orthogonal projection of $\mathbf{u} = (1, 0, 0, 0)$ onto the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1 = (1, 1, 1, 1)$, $\mathbf{v}_2 = (1, 1, -1, -1)$ and $\mathbf{v}_3 = (1, -1, -1, 1)$.
8. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Prove that the set
   $$\mathbf{v}^{\perp} = \{\mathbf{w} \in \mathbb{R}^n \mid \mathbf{w} \text{ is orthogonal to } \mathbf{v}\}$$
   is a subspace of $\mathbb{R}^n$ (called the *orthogonal subspace to* $\mathbf{v}$).
9. Consider the vector $\mathbf{v} = (1, 2, 3, 4)$ in $\mathbb{R}^4$. Find a basis of the subspace of $\mathbb{R}^4$ consisting of all vectors perpendicular to $\mathbf{v}$.
10. Five students took aptitude exams in English, mathematics and science. Their scores are shown below. Find the correlation coefficients r_{EM}, r_{ES} and r_{MS} between the three pairs of variables. Which ones are positively/negatively correlated?

Student	English	Math	Science
S1	61	53	53
S2	63	73	78
S3	78	61	82
S4	65	84	96
S5	63	59	71
Mean	66	66	76

11. Section 6.1 of the textbook, do Problems 4, 5, 6.
12. Section 6.2 of the textbook, do Problems 3, 5, 6, 7.

13. Let V be the space $\mathcal{C}[-1,1]$ of all continuous real-valued functions on the interval $[-1,1]$, and define an inner product on V by

$$\langle f, g\rangle \;=\; \int_{-1}^{1} f(x)g(x)\;dx. \tag{0.1}$$

Find a polynomial $p(x)$ of degree 2 that is orthogonal to 1 and x. Find a polynomial $q(x)$ of degree 3 that is orthogonal to 1, x and x^2. Are p and q orthogonal?

In the two problem, suppose that V is a vector space with an inner product and use the formulas below for the "angle" θ between two vectors $\mathbf{v}, \mathbf{w} \in V$ and the projection of $\mathbf{w}$ onto $\mathbf{v}$:

$$\cos\theta \;=\; \frac{\langle \mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{v}\|\cdot\|\mathbf{w}\|} \qquad\qquad \mathrm{Proj}_{\mathbf{v}}(\mathbf{w}) \;=\; \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|^2}\;\mathbf{v}.$$

15. Let V be the space $\mathcal{C}[0,4]$ with the L^2 inner product (like (0.1) above but integrating from 0 to 4).
    (a) What is the angle between the functions $f(x) = \frac{1}{2}x$ and $g(x) = \sqrt{x}$?
    (b) Decompose g as a vector parallel to f and one perpendicular to f, i.e. find a constant c and a function h with $g = af + h$ and $\langle f, h\rangle = 0$.

16. Similarly, let V be the space $\mathcal{C}[-\pi,\pi]$ with the L^2 inner product (integrate from $-\pi$ to π).
    (a) Find $\langle \sin x, \cos x\rangle$, $\langle \sin x, 1\rangle$ and $\langle \cos x, 1\rangle$ where 1 whose value at every x is 1.
    (b) What is the projection of the function $f(x) = x^2$ onto the subspace spanned by 1 and $\cos x$?

$$\textit{Hint}: \qquad \int x^2 \cos x\;dx = (x^2-2)\sin x + 2x\cos x = C.$$

17. In $\mathbb{R}^3$ with the euclidean inner product, the vectors $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (-2,1,1)$ and $\mathbf{v}_3 = (0,1,-1)$ are orthogonal (not orthonormal!). Use the inner product (i.e. the dot product) to write the vector $\mathbf{w} = (-3,7,2)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.