Math 415 Two Solutions

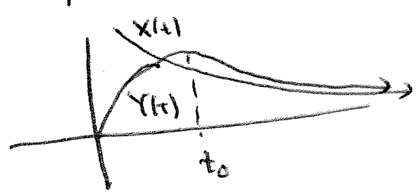
§5.2 Problem 6 The example in the text describes radioactive decay of element X to Y at rate r_1, then Y to Z at rate r_2. With initial condition $\binom{1}{0}$ the solution is

$$X(t) = e^{-r_1 t}$$
$$Y(t) = \frac{r_1}{r_2 - r_1}\left(e^{-r_1 t} - e^{-r_2 t}\right) \qquad (\text{eq. } 5.40)$$

By calculus, the max. of $Y(t)$ occurs when $Y'(t) = 0 \Rightarrow 0 = -r_1 e^{-r_1 t} + r_2 e^{-r_2 t}$

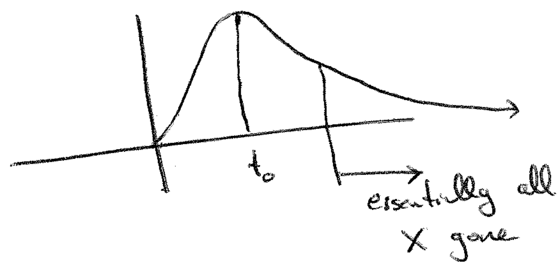
$$\Rightarrow e^{-r_1 t} = \frac{r_2}{r_1} e^{-r_2 t} \Rightarrow -r_1 t = \ln(r_2/r_1) - r_2 t$$
$$\Rightarrow t_{max} = \frac{1}{r_2 - r_1} \ln(r_2/r_1)$$

- If $r_2 > r_1$, X decays slowly to Y, then quickly to Z and for large t $Y(t) \sim c\, e^{-r_1 t}$

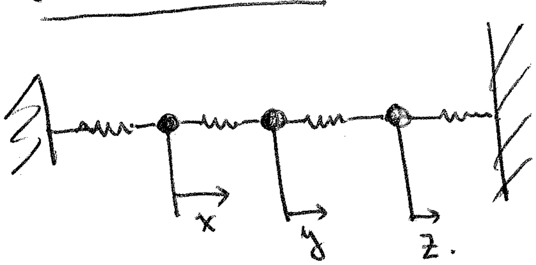


- If $r_2 < r_1$ X decays quickly to Y, then slowly to Z and

$Y(t) \sim C\, e^{-r_2 t}$



§5.3 Problem 5



	Displacements	Forces
Spring 1	x	$-kx$ on mass 1
Spring 2	$y-x$	$-k(y-x)$ on mass 2, $+k(y-x)$ on mass 1
Spring 3	$z-y$	$-k(z-y)$ on m3, $+k(z-y)$ on m2
Spring 4	z	$-kz$ on m3

Hence the system is

$$\begin{cases} m\dot{x} = -kx + k(y-x) = -2kx + ky. \\ m\dot{y} = -k(y-x) + k(z-y) = kx - 2ky + kz \\ m\dot{z} = -k(z-y) - kz = ky - 2kz \end{cases}$$

or $\dot{x} = Ax$ where $A = \frac{k}{m}\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$

Find e-values: $p(\lambda) = \begin{vmatrix} \lambda+2 & -1 & 0 \\ -1 & \lambda+2 & -1 \\ 0 & -1 & \lambda+2 \end{vmatrix} = (\lambda+2)[\lambda^2+4\lambda+3] - [\lambda+2]$

$= \lambda^3 + 6\lambda^2 + 10\lambda + 4.$

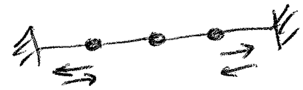
By trial-and-error one finds that $\lambda = -2$ is one root. Long division shows $p(\lambda) = (\lambda+2)(\lambda^2+4\lambda+2)$ so the e-values are $\underline{\lambda = [-2, -2 \pm \sqrt{2}]}$ all times k/m

$$\begin{array}{r} \lambda^2+4\lambda+2 \\ \lambda+2 \,\overline{)\, \lambda^3+6\lambda^2+10\lambda+4} \\ -\ \lambda^3+\ \lambda^2 \\ \hline 4\lambda^2+10\lambda \\ -\ 4\lambda^2+8\lambda \\ \hline 2\lambda+4 \end{array}$$

Finding the corresponding e-vectors yields 3 normal

Normal modes:

1) $\lambda = -\frac{2k}{m}$ and mode $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Middle mass stationary; outer masses oscillate in opposite directions with frequency $\omega = \sqrt{\frac{2k}{m}}$



2) $\lambda = (-2+\sqrt{2})k/m$ and mode $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$. All 3 masses oscillate together, with the middle mass moving most. with frequency $\omega = \sqrt{(2-\sqrt{2})\frac{k}{m}}$

3) $\lambda = (-2-\sqrt{2})k/m$ and mode $\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$. Two outer masses oscillate together with frequency $\omega = \sqrt{(2+\sqrt{2})k/m}$ (fastest), and the middle mass moves in opposite direction and $\sqrt{2}$ times as far.