Math 415

Homework Set 5

due Monday, Feb 20

All of the following problems are from Chapter 4 of the Sadun textbook.

1. Problem 1of Section 4.7.
2. Problem 2 of Section 4.7.
3. Problem 6 of Section 4.7.
4. Problem 7 of Section 4.7.
5. Compute e^{tA} for $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Use the standard Taylor series:

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \qquad \cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \qquad \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots .$$

6. Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$.
   (a) Find all eigenvalues and eigenvectors.
   (b) Find P such that $D = P^{-1}AP$ is diagonal. Also, find P^{-1}.
   (c) Find A^6 and $f(A)$ where $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 3$.
   (d) Find a "real cube root of A", i.e. a matrix B with real eigenvalues such that $B^3 = A$.
7. Problems 8 and 9 of Section 4.8.
8. (This problem is the same as Problems 1-3 in Section 4.9.) Consider a linear transformation $L : V \to V$. Recall that the *power set* of an eigenvalue λ of L is

$$\tilde{E}_\lambda \;=\; \big\{ v \in V \;\big|\; (L - \lambda I)^p v = 0 \text{ for some } p \big\}.$$

   (a) Show that L maps $\tilde{E}_\lambda$ to itself, that is, if $v \in \tilde{E}_\lambda$ then $Lv \in \tilde{E}_\lambda$.
   (b) Show that if $\mu \neq \lambda$ then $L - \mu I$ maps $\tilde{E}_\lambda$ to itself, ii.e. that $v \in \tilde{E}_\lambda \implies (L - \mu I)v \in \tilde{E}_\lambda$.
   (c) Prove that any collection $\{v_1, v_2, \ldots, v_k\}$ of power vectors corresponding to different eigenvectors is linearly independent.

Problem 6 in Section 4.2 If $b_1, \ldots b_n$ is a basis of eigenvectors of A with corresponding eigenvalues λ_i, show that the product $P = (A - \lambda_1 I) \cdots (A - \lambda_n I)$ equals the zero matrix.

Solution: P is the composition of linear transformations, so is itself a linear transformation. Hence it suffices to show that $Pb_i = 0$ for each i. But for each i and j, $(A - \lambda_j I)v_i = \lambda_i v_i - \lambda_j v_i = (\lambda_i - \lambda_j$. But then

$$Pb_i \;=\; (\lambda_i - \lambda_1)(\lambda_i - \lambda_2) \cdots (\lambda_i - \lambda_n) v_i \;=\; 0$$

because one factor in this product is equal to 0.