Math 415

Homework Set 4

due Monday, Feb 13

All of the following problems are from Chapter 4 of the Sadun textbook.

1. Problems 2, 3, and 4 of Section 4.1.
2. Problems 3, 4, 5, 7, 8 and 9 of Section 4.3.
3. Problems 4 and 6 in Section 4.2 (Hint for Problem 6: apply the product to an arbitrary vector v).
4. Problems 2, 3, 4, 7 and 11 in Section 4.4.

Algorithm for finding the inverse of an $n \times n$ matrix A.

1. Write down the augmented matrix $\big(A \,|\, I_n\big)$.
2. Do row operations to put A in reduced row echelon form, doing the same operations on the I_n side.
3. Is the reduced row echelon form of A is the identity?
   - If yes, the end result is $\big(I_n \,|\, B\big)$ where B is the matrix for A^{-1}.
   - If not, then A^{-1} does not exist, i.e. A is singular.

A *linear system* is a set of equations of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n} & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n} & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn} & = & b_m \end{array}$$

It is a *homogeneous linear system* of all of the b's are 0. To solve a linear system,

1. Write the system as an augmented matrix.
2. Apply Gaussian elimination (see notes with HW 2) to transform the augmented matrix to reduced row echelon form.

3. Read off the solution set by:
   - write down the corresponding system.
   - identify the pivot columns and the non-pivot variables.
   - Set each non-pivot variable to a free variable $r, s, t \ldots$ (e.g. $x_3 = r$); include these in the system.

For example, if the reduced augmented matrix is

$$\begin{bmatrix} 1 & 5 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 0 & 1 & 3 & 7 \end{bmatrix} \quad \text{then the system is} \quad \begin{aligned} x_1 + 5x_2 - x_5 &= 3 \\ x_3 + 4x_5 &= 5 \\ x_4 + x_5 &= 7 \end{aligned}$$

The non-pivot columns are the second (x_2) and the fifth (x_5), so include the equations $x_2 = r$ and $x_5 = s$.

The system is then $x_1 = 3 - 2r + s, x_2 = r, x_3 = 5 - 4s, x_4 = 7 - s$ and $x_5 = s$, so the solutions set is the set of all 5-tuples $(x_1, x_2, x_3, x_4, x_5)$ satisfying these equations, which we write as

$$S = \left\{(3 - 5r + s, r, 5 - 4s, 7 - s, s) \;\middle|\; r, s \in \mathbb{R}\right\}.$$

Alternatively, one can factor out the free variables and write

$$S = \left\{(3, 0, 5, 7, 0) + r(-5, 1, 0, 0, 0) + s(1, 0, -4, -1, 1) \;\middle|\; r, s \in \mathbb{R}\right\}.$$