Changing Bases

Suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ are two bases of a vector space V. Given a vector $\mathbf{v} \in V$, we can write $\mathbf{v} = \sum x^i e_i$ and form the column vector

$$\mathbf{v}_{\mathbf{e}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{\mathbf{e}}$$

This is the *coordinate vector of* $\mathbf{v}$ *in the basis* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. We can write

$$\mathbf{v} \underset{\mathbf{e}}{\longleftrightarrow} \mathbf{v}_{\mathbf{e}}$$

When the same vector $\mathbf{v}$ is expanded in the basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ it has different coefficients

$$\mathbf{v} \underset{\mathbf{f}}{\longleftrightarrow} \mathbf{v}_{\mathbf{f}} \;=\; \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}_{\mathbf{f}}$$

How are the y's related to the x's?

Finding the coordinates in a new basis: *The coordinate vectors* $\mathbf{v}_{\mathbf{e}}$ *and* $\mathbf{v}_{\mathbf{f}}$ *are related by*

$$\mathbf{v}_f \;=\; P_{\mathbf{fe}}\mathbf{v}_{\mathbf{e}}$$

Note how the subscripts are arranged.

where $P_{\mathbf{fe}}$ and its inverse $P_{\mathbf{ef}} = P_{\mathbf{fe}}^{-1}$ are the **change-of-basis matrices** defined by

$$P_{\mathbf{ef}} \;=\; \text{the matrix whose } i^{th} \text{ column is } [\mathbf{f}_i]_e.$$

Finding the matrix in a new basis: *If a linear transformation* $L : V \to V$ *has matrix* $A_{\mathbf{ee}}$ *in the basis* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$*, then its matrix in the basis* $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ *is*

$$A_{\mathbf{ff}} \;=\; P_{\mathbf{fe}}A_{\mathbf{ee}}P_{\mathbf{ef}} \;=\; P_{\mathbf{fe}}^{-1}A_{\mathbf{ee}}P_{\mathbf{fe}}.$$

Note how the subscripts are arranged.

Example. (i) Let $\mathbf{v}$ be the vector whose coordinates are $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$. What are its coordinates in the basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ where $\mathbf{f}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{f}_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$?

(ii) If $L : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\begin{pmatrix} 2 & 3 \\ 1 & 7 \end{pmatrix}$ in the standard basis, what is it in the basis $\{\mathbf{f}_1, \mathbf{f}_2\}$?

Solution: $P_{\mathbf{ef}}$ is the matrix whose columns are the $\mathbf{f}$'s, so $P_{\mathbf{ef}} = \begin{pmatrix} 3 & -3 \\ 1 & 4 \end{pmatrix}$. Then $P_{\mathbf{fe}}$ is its inverse, so (use our trick for writing down the inverse of a 2×2 matrix) $P_{\mathbf{fe}} = \frac{1}{15}\begin{pmatrix} 4 & 3 \\ -1 & 3 \end{pmatrix}$. Then

(i) $$P_{\mathbf{fe}}\mathbf{v}_e \;=\; = \;\frac{1}{15}\begin{pmatrix} 4 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} \;=\; \frac{1}{15}\begin{pmatrix} 14 \\ -11 \end{pmatrix}$$

(ii) $$P_{\mathbf{fe}}^{-1}AP_{\mathbf{fe}} \;=\; \begin{pmatrix} 3 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 7 \end{pmatrix}\frac{1}{15}\begin{pmatrix} 4 & 3 \\ -1 & 3 \end{pmatrix} \;=\; \frac{1}{15}\begin{pmatrix} 4 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 9 & 6 \\ 10 & 25 \end{pmatrix} \;=\; \frac{1}{15}\begin{pmatrix} 24 & -27 \\ -7 & 111 \end{pmatrix}$$

Math 415

Homework Set 3

due Friday, Feb 3

1. Do Problems 15, 16, 17, and 18 on page 42 of Sadun's book.

2. Use Gaussian Elimination to solve the following two linear systems (ie do row operations on the augmented matrix).

$$(a)\quad \begin{aligned} x_1 - x_2 - x_3 &= -1 \\ -2x_1 + 6x_2 + 10x_3 &= 14 \\ 2x_1 + x_2 + 6x_3 &= 9 \end{aligned} \qquad (b)\quad \begin{aligned} x_1 + 2x_2 + 2x_3 + 5x_4 &= 11 \\ 2x_1 + 4x_2 + 2x_3 + 8x_4 &= 14 \\ x_1 + 3x_2 + 4x_3 + 8x_4 &= 19 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}$$

3. In this exercise, you are given a basis $\mathbf{f} = \{\mathbf{f}_1, \mathbf{f}_2\}$ for $\mathbb{R}^2$ (or for $\mathbb{R}^3$) and the $\mathbf{e}$-coordinates of a vector $\mathbf{v}$. Find $P_{\mathbf{ef}}, P_{\mathbf{fe}}$ and $\mathbf{v}_f$. *Follow the method described on the back of this page.*

   (a) $\mathbf{f}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{f}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, and $\mathbf{v}_\mathbf{e} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

   (b) $\mathbf{f}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\mathbf{f}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, and $\mathbf{v}_\mathbf{e} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$.

   (c) $\mathbf{f}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{f}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{f}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and $\mathbf{v}_\mathbf{e} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$.

4. Do Problem 3 in Section 3.3.

5. Do Problem 7 in Section 3.3.

6. Read Section 3.4 in the textbook (on infinite dimensional matrices) and do Problems 1, 2, and 3 in that section.

   Note: Problem 3 asks you to first find the composition $L_1 \circ L_2$ of the linear transformations in Problems 1 and 2 (as transformations, not matrices) in both orders. Then is asks you to compare with the products of the matrices.