Math 415

Homework Set 2

due Friday, Jan 27

1. Let $V = C^{\infty}(\mathbb{R})$ be the vector space of all functions that have derivatives of all orders. Let $D : V \to V$ be the derivative (i.e. $Df = \frac{df}{dx}$).

   (a) What is the kernel of D?

   (b) Let D^2 be the second derivative. What is the kernel of D^2?

   (c) What is the kernel of $D^n : V \to V$?

2. With V and D as above, let $L = D - I$ where I is the identity transformation of V to V.

   (a) What is $\ker L$? Answer by completing the sentence: $\ker L$ is the set of all functions in V such that ....

   (b) Same question for $L = D - aI$ for any real number a.

3. Let $L : V \to W$ be a linear transformation. Fix a vector in w_0 and think about solving the equation $Lv = w_0$ for the unknown v. Suppose that v_0 is one solution. Show that every other solution is of the form $v = v_0 + u$ where $u \in \ker L$.

4. Let V and D be as in Problem 1 and let g by an arbitrary element of V. Explain how the problem of solving the differential equation

$$a\frac{d^2 f}{dx^2} + b\frac{df}{dx} + cf \;=\; g$$

   fits into the abstract situation described in Problem 3.

The next 5 problems are Problems 6-10 for Section 2.3 in Sadun's book. **Notation:** $\mathbb{R}_d[t]$ denotes the vector space of all polynomials of degree $\leq d$ in the variable t with real coefficients, and $M_{m,n}$ denotes the vector space of all $m \times n$ matrices with real entries.

5. Let $V = \mathbb{R}_2[t]$. Determine whether the vectors $\mathbf{b}_1 = 1 + t + t^2$, $\mathbf{b}_2 = 1 + 2t + 3t^2$, $\mathbf{b}_3 = 1 + 4t + 9t^2$ are linearly independent, span V, and/or are a basis.

6. Let V be the subspace of $\mathbb{R}_3[t]$ consisting of all polynomials $\mathbf{p}$ with $\mathbf{p}(0) = 0$. Determine whether the vectors

$$\mathbf{b}_1 = t^2 - t \qquad \mathbf{b}_2 = t^3 + t^2 + t \qquad \mathbf{b}_3 = 2t^3 - 5t^2 - 7t$$

   are linearly independent, span V, and/or are a basis.

7. Let V be the subspace of $\mathbb{R}_3[t]$ consisting of all polynomials $\mathbf{p}$ with $\mathbf{p}(1) = 0$. Determine whether the vectors

$$\mathbf{b}_1 = t^2 - t \qquad \mathbf{b}_2 = t^3 + t^2 + t - 3 \qquad \mathbf{b}_3 = 2t^3 - 5t^2 - 7t + 10$$

   are linearly independent, span V, and/or are a basis.

8. In $M_{2,2}$, are the matrices $\mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{A}_2 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$, $\mathbf{A}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ linear independent, a spanning set, and/or a basis?

9. Let V be the subspace of $M_{2,2}$ consisting of all symmetric matrices, i.e. those of the form

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

What is the dimension of V? Find a basis.

The remaining problems are taken from Section 3.2 of Sadun's book and Section 3 of Treil's book (your class notes form Monday should allow you to do these).

10. For each linear transformation below find its matrix.
    (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$.
    (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x-5y \\ 7y \end{pmatrix}$.
    (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ by counterclockwise rotation through 45 degrees.

11. Find 3×3 matrices representing the transformations of $\mathbb{R}^3$ which:
    (a) project every vector to the xy plane,
    (b) reflect every vector through the xy plane
    (c) rotate the xy plane counterclockwise through $30°$, leaving the z-axis fixed.

12. Let $V = \mathbb{R}_3[t]$ and let $D : V \to V$ be the differentiation transformation $Df = \frac{df}{dt}$. Using the standard basis $\{1, t, t^2, t^3\}$ of V (in that order!),
    (a) Write down the matrix for D (done in class and in Sadun's Section 3.2).
    (b) Write down the matrix for the second derivative transformation $L(f) = \frac{d^2 f}{dt^2}$. Show by matrix multiplication that $L = D^2$.
    (c) Use your answers to parts (a) and (b) to write down the matrix for the transformation T defined by

$$(Tf)(t) \;=\; 2f(t) + 3f'(t) - 4f''(t).$$