Math 415

Bonus Homework Set

due Monday, April 23

These problems are on dual spaces.

Warm-up: Let V be a 3-dimensional real vector space with basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Each vector $\mathbf{v} \in V$ can be written as $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Which of the following functions on V are linear functions?

1. $\phi(\mathbf{v}) = x + y + z$.
2. $\phi(\mathbf{v}) = (x + y)^2$.
3. $\phi(\mathbf{v}) = \sqrt{2}\,x$
4. $\phi(\mathbf{v}) = y - \frac{1}{2}z$.
5. $\phi(\mathbf{v}) = z - \frac{1}{2}$.

1. Let $V = P_n$ be the space of polynomials of degree at most n with real coefficients. For each fixed $c \in \mathbb{R}$, let $\phi(p) = p''(c)$ (i.e. the value of the second derivative of p at $x = c$). Show that ϕ is a linear functional.
2. Let $V = C([0, 1])$ be the space of continuous real-valued functions on the interval $[0, 1]$. Fix a function $g \in V$. For each $f \in V$, define

   $$\phi_g(f) \;=\; \int_0^1 f(x)g(x)\,dx.$$

   (a) Show that ϕ_g is a linear functional on V.

   (b) Show that if $\phi_g(f) = 0$ for every $g \in V$ then $f = 0$.
3. Let V be a finite dimensional vector space of dimension $n \geq 2$.

   (a) Suppose that $\mathbf{v}, \mathbf{w}$ are two linearly independent vectors in V. Show that there is a linear functional ϕ such that $\phi(\mathbf{v}) = 1$ and $\phi(\mathbf{w}) = 0$. *Begin by expanding* $\{\mathbf{v}, \mathbf{w}\}$ *to a basis.*

   (b) Similarly, let W be a subspace of V and $\mathbf{v} \in V$ a vector that is NOT in W. Show that there is a linear functional ϕ such that $\phi(\mathbf{v}) = 1$ and $\phi(\mathbf{w}) = 0$ for all $\mathbf{w} \in W$.