**Math 415**

# Homework Set 11

due Monday, April 16

The first three problems are Problems 1,2 and 8 in Section 7.3 of the textbook.

1. Let $f(\mathbf{x}) = 6x_1^2 + 4x_1x_2 + 3x_2^2$. Write $f$ as a function of $\theta$ by setting $x_1 = \cos\theta$ and $x_2 = \sin\theta$ and find all maxima and minima of $f$. How many of each are there?

2. With $f(\mathbf{x})$ as in Problem 1, show that the curve $f(\mathbf{x}) = 1$ is an ellipse. What are the semi-major and semi-minor axes?

3. Consider the quadratic function $Q(\mathbf{x}) = x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2$ on $\mathbb{R}^3$. Restrict this function to the unit sphere $S = \{|\mathbf{x}| = 1\}$. Find the maxima, minima and saddle points.

   **Hint:** The critical points of $Q(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x}$ are found by finding the eigenvalues and eigenvectors of $A$!

   *Answer from back of the book: The maximum value of* $1 + \sqrt{2}/2$ *is achieved at* $\pm\frac{1}{2}(1, \sqrt{2}, 1)^T$*, the minimum value of* $1 - \sqrt{2}/2$ *is achieved at* $\pm\frac{1}{2}(1, -\sqrt{2}, 1)^T$*, and saddle points are* $\pm\frac{1}{\sqrt{2}}(1, 0, -1)^T$.

4. Do Problem 11 in Section 7.4.

5. Do Problem 1 in Section 7.5 (the marices $K_1, K_2, K_3$ are given on the previous page).

6. Do Problem 4 in Section 7.5. *This is easy. Start by asking which complex matrices* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *satisfy* $\overline{A}^t = A$.

7. Do Problem 6 in Section 7.5. *Use power series, just as you did for Problem 5 in HW Set 5.*

8. Let $A$ be an $n \times n$ complex matrix.

   (a) Use power series to show that $e^{\bar{A}} = \overline{e^A}$.

   (b) Prove that if $A$ is skew-hermitian (i.e. if $\overline{A}^t = -A$), then $B = e^{tA}$ is unitary for all $t$ (i.e. $\overline{B}^t = B^{-1}$).

## Solution to HW10 Problem 4

The function $\psi_r$ is defined by $\psi_r(t) = \psi(t - t_r)$ where $t_r = \frac{r}{(2N+1)}$ and

$$\psi(t) = \frac{1}{2N+1}\Big((1 + 2\sum_j \cos(2\pi jt)\Big).$$

Hence

$$(2N+1)^2 \int_{\frac{1}{2}}^{\frac{1}{2}} \psi_r(t)\psi_s(t)\; dt \;\; = \;\; \int_{\frac{1}{2}}^{\frac{1}{2}} (1 + 2\sum_j \cos(2\pi j(t - t_r)) \cdot (1 + 2\sum_k \cos(2\pi k(t - t_s))$$

This has the form $(1 + A)(1 + B) = 1 + A + B + AB$. The integrals of $A$ and $B$ vanish because the integral of $\cos(2\pi j(t - t_s))$ over a complete period is 0. Expand the $AB$ term using $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ for both $A$ and $B$. Noting that $c_{rj} = \cos(2\pi jt_r)$ and $d_{rj} = \sin(2\pi jt_r)$ are constants, the above integral is:

$$1+4\sum_{jk} \int_{\frac{1}{2}}^{\frac{1}{2}} c_{rj}c_{sk}\cos(2\pi jt)\cos(2\pi kt) + (c_{rj}d_{sk}+d_{rj}c_{sk})\cos(2\pi jt)\sin(2\pi jt) + d_{rj}d_{sk}\sin(2\pi jt)\sin(2\pi kt)$$

But the functions $\cos(2\pi jt)$ and $\sin(2\pi jt)$ are $L^2$ perpendicular so, after integrating, the middle term vanishes. Similarly, the first and third terms vanish unless $j = k$, in which case they can be evaluated using the fact that the average value of $\sin^2$ and $\cos^2$ over a complete period is $\frac{1}{2}$. The result is

$$1 + 2\sum_j c_{rj}c_{sj} + d_{rj}d_{sj} \;\; = \;\; 1 + 2\sum_{jk} \cos(2\pi j(t_r - t_s))$$

(again using the trig identity). But this last expression is exactly $(2N+1)\psi_s(t_r)$ and we know that $\psi_s(t_r) = \delta_{rs}$ (as stated after equation (4.4) in the handout). Thus, altogether, we have

$$\int_{\frac{1}{2}}^{\frac{1}{2}} \psi_r(t)\psi_s(t)\; dt \;\; = \;\; \frac{1}{2N+1}\;\delta_{rs}.$$