Lemma 4c Suppose $\alpha\bar{v}=\bar{0}$, and $\alpha \neq 0$.

Then $\frac{1}{\alpha}$ exists (as a real number),

$\alpha \bar{v} = \bar{0}$ (given)

$\frac{1}{\alpha} \cdot (\alpha \bar{v}) = \frac{1}{\alpha} \cdot \bar{0}$ (prop. =)

$\left(\frac{1}{\alpha} \cdot \alpha\right) \bar{v} = \bar{0}$ (A7 on l.h.s., Lemma 4B on r.h.s.)

$1 \cdot \bar{v} = \bar{0}$ (prop. $\mathbb{R}$: $\alpha \cdot \frac{1}{\alpha} = 1$)

$\bar{v} = \bar{0}$ (A8)

So, if $\alpha \neq 0$, then $\bar{v} = \bar{0}$ which completes the proof. $\blacksquare$

Lemma 5 $\bar{x} + \bar{v} = \bar{y} + \bar{v}$ (given)

$(\bar{x} + \bar{v}) + (-\bar{v}) = (\bar{y} + \bar{v}) + (-\bar{v})$ (prop. =)

$\bar{x} + (\bar{v} + (-\bar{v})) = \bar{y} + (\bar{v} + (-\bar{v}))$ (A2 both sides)

$\bar{x} + \bar{0} = \bar{y} + \bar{0}$ (A4 for each $\bar{v} + (-\bar{v})$)

$\bar{x} = \bar{y}$ (A3 both sides)

$\blacksquare$

Lemma 6 $\bar{v} + \bar{x} = \bar{0}$ (given)

$(-\bar{v}) + (\bar{v} + \bar{x}) = (-\bar{v}) + \bar{0}$ (prop. =, add $(-\bar{v})$)

$((-\bar{v}) + \bar{v}) + \bar{x} = -\bar{v}$ (A2 on l.h.s., A3 on r.h.s.)

$(\bar{v} + (-\bar{v})) + \bar{x} = -\bar{v}$ (A1 inside brackets)

$\bar{0} + \bar{x} = -\bar{v}$ (A4 inside brackets)

$\bar{x} + \bar{0} = -\bar{v}$ (A1 on l.h.s.)

$\bar{x} = -\bar{v}$

Lemma 6.5: $\bar{v} + (-1)\cdot\bar{v} = \bar{0}$

Proof: $\bar{v} + (-1)\bar{v} = 1\cdot\bar{v} + (-1)\cdot\bar{v}$ (A3)

$\bar{v} + (-1)\bar{v} = (1 + (-1))\cdot\bar{v}$ (A6 to r.h.s.)

$\bar{v} + (-1)\cdot\bar{v} = 0\cdot\bar{v}$ (prop. $\mathbb{R}$: $1+(-1)=0$)

$\bar{v} + (-1)\cdot\bar{v} = \bar{0}$ (Lemma 4a)

$\blacksquare$

Lemma 7 Let $\bar{x} = (-1) \cdot \bar{v}$. By Lemma 6.5,

$\bar{v} + \bar{x} = 0$. By Lemma 6, $\bar{x} = -\bar{v}$.

Hence, $(-1)\bar{v} = -\bar{v}$.

Lemma 8 $\bar{v} = -\bar{v}$ (given)

$\bar{v} + \bar{v} = \bar{v} + (-\bar{v})$ (prop. =, add $\bar{v}$)

$1 \cdot \bar{v} + 1 \cdot \bar{v} = \bar{0}$ (A3 on l.h.s., A4 on r.h.s.)

$(1+1)\bar{v} = \bar{0}$ (A6 on l.h.s.)

$2\bar{v} = \bar{0}$ (prop. $\mathbb{R}$: $1+1=2$)

$\frac{1}{2} \cdot (2\bar{v}) = \frac{1}{2} \cdot \bar{0}$ (prop. = : multiply by $\frac{1}{2}$)

$\left(\frac{1}{2} \cdot 2\right)\bar{v} = \bar{0}$ (A7 on l.h.s., Lemma 4B on r.h.s.)

$1 \cdot \bar{v} = \bar{0}$ (prop. $\mathbb{R}$: $\frac{1}{2} \cdot 2 = 1$)

$\bar{v} = \bar{0}$ (A 8)

$\blacksquare$

Lemma 9

$$-(-\bar{v}) = (-1)\cdot(-\bar{v}) \quad \text{(Lemma 7)}$$

$$-(-\bar{v}) = (-1)\cdot((-1)\cdot\bar{v}) \quad \text{(Lemma 7 again in r.h.s.)}$$

$$-(-\bar{v}) = ((-1)\cdot(-1))\cdot\bar{v} \quad \text{(A7)}$$

$$-(-\bar{v}) = 1\cdot\bar{v} \quad (\text{prop. } \mathbb{R}: (-1)\cdot(-1)=1)$$

$$-(-\bar{v}) = \bar{v} \quad \text{(A8)}$$

Lemma 10

$$\alpha(-\bar{v}) = \alpha((-1)\cdot\bar{v}) \quad \text{(Lemma 7)}$$

$$\alpha(-\bar{v}) = (\alpha\cdot(-1))\cdot\bar{v} \quad \text{(A7 in r.h.s.)}$$

$$(*) \quad \alpha\cdot(-\bar{v}) = ((-1)\cdot\alpha)\cdot\bar{v} \quad (\text{prop. } \mathbb{R}: (-1)\cdot\alpha = \alpha\cdot(-1))$$

$$\alpha\cdot(-\bar{v}) = (-1)\cdot(\alpha\bar{v}) \quad \text{(A7 in r.h.s.)}$$

$$\alpha\cdot(-\bar{v}) = -(\alpha\bar{v}) \quad \text{(Lemma 7 to r.h.s.)}$$

Lemma 11

$$(-\alpha)\bar{v} = ((-1)\cdot\alpha)\bar{v} \quad (\text{prop. } \mathbb{R}: -\alpha = (-1)\cdot\alpha)$$

Then repeat transsformations from Lemma 10 starting from $(*)$ in the r.h.s.

Lemma 12 $\alpha\bar{v} = \beta\bar{v}$ (given)

$$\alpha\bar{v} + (-(\beta\bar{v})) = \beta\bar{v} + (-(\beta\bar{v}))$$ (prop. =/ add $(-(\beta\bar{v}))$)

$$\alpha\bar{v} + (-(\beta\bar{v})) = \bar{0}$$ (A3 in r.h.s.)

$$\alpha\bar{v} + (-\beta)\bar{v} = \bar{0}$$ (Lemma 11 to the r.h.s.)

$$(\alpha + (-\beta))\bar{v} = \bar{0}$$ (A6 to l.h.s.)

$$(\alpha - \beta)\bar{v} = \bar{0}$$ (prop. $\mathbb{R}$: $\alpha + (-\beta) = \alpha - \beta$)

$$\left.\begin{array}{l}(\alpha-\beta)\bar{v} = \bar{0} \\ \bar{v} \neq \bar{0} \text{ (given)}\end{array}\right\} \Longrightarrow \alpha - \beta = 0$$ (Lemma 4c)

$$\Downarrow$$

$$\alpha = \beta$$ (prop. $\mathbb{R}$)
