Sec. 6.1 Eigenvalues and Eigenvectors

Linear transformations $L : V \to V$ that go from a vector space to itself are often called linear *operators*. Many linear operators can be understood geometrically by identifying directions in which T acts as a dilation: for each vector $\mathbf{v}$ in these directions, $T(\mathbf{v})$ is a multiple of $\mathbf{v}$. Such vectors $\mathbf{v}$ are called *eigenvectors* and the corresponding multiples are called *eigenvalues*. Chapter 5 explains how to find eigenvalues and eigenvectors, and how to use them to understand linear operators.

Here is the main definition (memorize this!)

Definition. *Let $L : V \to V$ be a linear operator. A scalar λ ("lambda") is called an* **eigenvalue** *of L if there is a non-zero vector $\mathbf{v}$ in V such that*

$$L(\mathbf{v}) \;=\; \lambda\mathbf{v}.$$

The vector $\mathbf{v}$ is called an **eigenvector** *of T corresponding to the eigenvalue λ.*

Each $n \times n$ matrix A specifies an operator $A : \mathbb{R}^n \to \mathbb{R}^n$, so we can express the above definition in terms of the matrix: A scalar λ is an **eigenvalue** of A if there is a non-zero $\mathbf{v} \in \mathbb{R}^n$ such that

$$A\mathbf{v} \;=\; \lambda\mathbf{v}.$$

The vector $\mathbf{v}$ is called an **eigenvector** of A corresponding to λ.

Finding Eigenvalues. To solve the equation $A\mathbf{v} = \lambda\mathbf{v}$, we treat both $\mathbf{v}$ and λ as unknowns — think of λ as a variable much like one treats x in high school algebra. The key fact is:

Theorem 5.2 *Let A be an $n\times n$ matrix. Then a scalar λ is an eigenvalue of A if and only if* $\det(A - \lambda I_n) = 0$.

Proof.

$$\begin{aligned} A\mathbf{v} \;=\; \lambda\mathbf{v} \quad &\iff \quad (A-\lambda I)\mathbf{v} = 0 \\ &\iff \quad (A-\lambda I) \text{ is singular} \\ &\iff \quad \det(A-\lambda I) = 0. \end{aligned}$$

□

Definition. The **characteristic polynomial** of a square matrix A is

$$p_A(\lambda) = \det(A - \lambda I)$$

If A is an $n \times n$ matrix, $p_A(\lambda)$ is a polynomial in λ of the form

$$p_A(\lambda) \;=\; (-1)^n\lambda^n \;+\; \text{lower degree terms.}$$

Note that λ is an eigenvalue if and only if there is a non-zero $\mathbf{v}$ with $(A - \lambda I)\mathbf{v} = 0$, and this occurs if and only if the matrix $A - \lambda I$ is singular, i.e. $\det(A - \lambda I) = 0$. Thus we have:

Theorem. *The eigenvalues of A are precisely the roots of the characteristic polynomial.*

This gives a way of finding the eigenvalues:

Finding Eigenvectors. Looking at the eigenvalue λ_i one at a time, one solves the equation $A\mathbf{v} = \lambda\mathbf{v}$ for each. The set of solutions is a subspace:

Definition. For each eigenvalue λ of a linear operator $L : V \to V$, the corresponding **eigenspace** is the set of all eigenvectors for with eigenvalue λ:

$$E_\lambda(L) \;=\; \left\{\mathbf{v} \in V \;\middle|\; L(\mathbf{v}) = \lambda\mathbf{v}\right\}.$$

This is a subspace for each λ (see Problem 1 below).

Method for finding eigenvalues and eigenspaces: Given a square matrix A,

1. Find the characteristic polynomial by writing down the matrix $A - \lambda I$ ($= A$ with λ subtracted from each diagonal entry), and calculating the determinant (and changing sign if helpful).
2. Factor the characteristic polynomial as much as possible; each linear factor $(\lambda - c)$ then gives an eigenvalue c.
3. For each eigenvalue λ_i, the eigenspace $E_A(\lambda_i)$ is the set of solutions to $(A - \lambda_i I)\mathbf{v} = \mathbf{0}$. Solve this system by Gaussian elimination or by inspection.

Caution: For some matrices A, the characteristic polynomial has no linear factors, so there are no eigenvalues.

Homework I — Due Wednesday Nov. 28.

Look over Section 6.1 of the textbook through the end of Example 4.

1. Let $L : V \to V$ be a linear operator. Using the definition of $E_\lambda(L)$ above, prove that $E_\lambda(L)$ is a subspace (not just a subset) of V.
2. (a) Explain why $E_0(L)$ is the null space $N(L)$ of L (which is also called $\ker L$).
   (b) Similarly explain why $E_\lambda(L)$ is the null space $N(L - \lambda I)$ of the linear operator $L - \lambda I$.
3. The matrix $A = \begin{pmatrix} 1 & 2 \\ 5 & -2 \end{pmatrix}$ has $\lambda = 3$ as an eigenvalue. Find a corresponding eigenvector.
4. Let $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Find a basis for $E_2(B)$ and a basis for $E_{-1}(B)$.
5. Let $L : P_5 \to P_5$ by $L = \frac{d^2}{dx^2}$.
   (a) Find a basis for $E_0(L)$. *Hint: elements of $E_0(L)$ are polynomials that satisfy what equation?*
   (b) Show that L has no non-zero eigenvalues.
6. Find all of the eigenvalues of the following matrices.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 6 & -24 & -4 \\ 2 & -10 & -2 \\ 1 & 4 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & -7 & -4 \\ -1 & 9 & 4 \\ 2 & -14 & -6 \end{pmatrix} \qquad D = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

7. Find all eigenvalues and eigenvectors of the following matrices.

$$E = \begin{pmatrix} -2 & 7 \\ 0 & 3 \end{pmatrix} \qquad F = \begin{pmatrix} 5 & -4 & 3 \\ 0 & -1 & 9 \\ 0 & 0 & 1 \end{pmatrix}$$

   (e) What feature of these matrices makes it easy to find their eigenvalues?

8. Write down a 2×2 matrix with eigenvalues 2 and 5. Make your matrix as simple as possible.

9. (a) Show that any symmetric 2×2 matrix, that is one of the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ has eigenvalues.

   (b) Under what condition on a and b will the matrix have two distinct eigenvalues?

10. Prove that a square matrix is singular if and only if 0 is one of its eigenvalues.

Sec. 6.3A Similarity and Diagonalizability

Let V be a vector space with basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Then each linear operator $L : V \to V$ can be written as a matrix $A = [L]_{EE}$ and, going backwards, each $n \times n$ matrix A determines a linear operator $L : V \to V$. Furthermore, if $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is another basis, then the matrix $B = [L]_{UU}$ of L in this β basis is

$$B = Q^{-1}AQ \tag{2.1}$$

where $Q = [L]_{EU}$ is the change-of-basis matrix (whose ith column is the coordinates of $\mathbf{u}_i$ in the basis E). This formula motivates the following definition.

Definition. *Two $n \times n$ matrices A and B are* **similar** *if there is an invertible $n \times n$ matrix Q such that $B = Q^{-1}AQ$.*

Proposition 1. Two similar matrices have

(a) the same trace,

(b) the same determinant,

(c) the same characteristic polynomial, and

(d) the same eigenvalues.

Proof. As shown in class, statements (b)-(d) follow from the fact that $\det(AB) = \det A \cdot \det B$. □

Example 1. The matrices $A = \begin{pmatrix} 2 & -4 \\ 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 1 \\ 4 & 2 \end{pmatrix}$ are not similar because $\det A = 22$, while $\det B = 10$.

It is natural to ask whether a given matrix is similar to one that has an especially simple form, specifically whether it is similar to a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

Definition. *A square matrix A is* **diagonalizable** *if it is similar to a diagonal matrix.*

Lemma 2. *Let $L : V \to V$ be a linear operator. Then any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of non-zero eigenvectors of L with distinct eigenvalues are linearly independent.*

Proof. This is proved on page 307 of the textbook, and was also done in class. □

Theorem 3. *If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.*

Proof. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be the linear operator whose matrix in the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is A. The hypothesis that A has distinct eigenvalues means that there are vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and constants $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, which are all different, so that $L\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for each i. By Lemma 2, $F = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a set of n linearly independent vectors in an n-dimensional space, so is a basis. Since L multiplies each $\mathbf{v}_i$ by λ_i, the matrix of L in the β basis is

$$D \;=\; [L]_{FF} \;=\; \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

The change-of-basis formula (2.1) above then shows that $D = Q^{-1}AQ$, so A is diagonalizable.
□

Procedure: To diagonalize an $n \times n$ matrix A with distinct eigenvalues:

1. Find the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$.
2. Factor as much as possible to find the eigenvalues λ_i (check the discriminant $b^2 - 4ac$ to see if quadratics factor).

If the eigenvalues are distinct, then A is diagonalizable. To diagonalize:

3. Find an eigenvector for each eigenvalue λ_i.
4. Write down the matrix Q whose columns are your basis eigenvectors.
5. Compute Q^{-1}.
6. Then $D = Q^{-1}AQ$ is the diagonal matrix with the eigenvalues down the diagonal.

Notice that *you can find D without first finding Q* because

$$D \;=\; \text{a diagonal matrix with the eigenvalues } \lambda_i \text{ (found in Step 2) on the diagonal.}$$

Furthermore, Homework Problem 4 below shows that the order of the λ_i doesn't matter: if A is similar to the diagonal matrix D with λs down the diagonal in one order, it is also similar to the diagonal matrix D' with the same λs on the diagonal in any other order.

Example 2. Diagonalize $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$.

The characteristic polynomial is

$$p(\lambda) = \det \begin{pmatrix} 5-\lambda & 6 \\ -2 & -2-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2),$$

so the eigenvalues are $\lambda = 1, 2$. These are different, so A is diagonalizable.

Next solve $A\mathbf{v} = \lambda\mathbf{v}$ in the form $(A - \lambda I)\mathbf{v} = 0$ using augmented matrices in the two cases:

- For $\lambda = 1$, solve $(A - I)\mathbf{v} = 0$ $\begin{pmatrix} 4-\lambda & 6 & 0 \\ -2 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ so one solution is $\mathbf{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.
- For $\lambda = 2$, solve $(A - 2I)\mathbf{v} = 0$ $\begin{pmatrix} 3 & 6 & 0 \\ -2 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ so one solution is $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Q is the matrix whose columns are these eigenvectors, and we can write down Q^{-1} by the usual trick:

$$Q = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \qquad\qquad Q^{-1} = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}$$

and then $D = Q^{-1}AQ = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is diagonal.

Homework II — Due Friday, Nov 30.

1. Find the eigenvalues and the corresponding eigenvectors for the matrices:

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}$$

(these are Problems 1(a), (h) from Section 6.1 of the textbook).

2. Prove that a scalar λ is an eigenvalue of an invertible linear operator L if and only if λ^{-1} is an eigenvalue of L^{-1}.
3. For the matrix $C = \begin{pmatrix} 3 & 2 \\ 6 & -1 \end{pmatrix}$, what is $\operatorname{tr} C$ and $\det C$, and what are the eigenvalues of C?
4. Show that $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is similar to $B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$. *Hint: write down the change-of-basis matrix Q that changes the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{e}_2, \mathbf{e}_1\}$.*
5. For the matrix $A = \begin{pmatrix} 2 & -3 \\ 2 & -5 \end{pmatrix}$ follow Example 2 above to find:
   (a) Find the characteristic polynomial and the eigenvalues.
   (b) Find eigenvectors for each eigenvalue.
   (c) Write down a matrix Q so that $Q^{-1}AQ$ is diagonal.
   (d) Find Q^{-1} and explicitly calculate $Q^{-1}AQ$ to show that it is diagonal.
6. Do the same for the matrix $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$ in Problem 1b of Section 6.3 of the textbook.
7. Do the same for the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix}$ in Problem 1e on the same page.

Sec. 6.3B Applications of Diagonalization

For an $n \times n$ matrix A, we have two facts and a consequence – these were discussed in the previous class and are proved on page 307 of the textbook:

Theorem. *(a) A is diagonalizable $\Leftrightarrow$ there is a basis of eigenvectors of A.*

(b) Any set of non-zero eigenvectors of L with distinct eigenvalues is linearly independent.

(c) Therefore, if A has n distinct eigenvalues, then A is diagonalizable.

On the other hand, some matrices are not diagonalizable.

Definition. *An $n \times n$ matrix A is called* defective *if it has fewer than n linearly independent eigenvectors (and hence is not diagonalizable.*

Examples. There are three possibilities:

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Here, A has 2 distinct eigenvalues (namely, one can check, $\lambda = 1, 2$), hence is diagonalizable (Good), B has $< n$ distinct eigenvalues, but is still is diagonalizable (OK), and C is defective (Bad).

Many matrix computations become easier if one can diagonalize the matrix. Geometrically, this means thinking of the matrix as a linear transformation and switching to a basis in which the linear transformation is a dilation in each direction.

Suppose that A is an $n \times n$ matrix that can be diagonalized. This means that there is an $n \times n$ matrix Q so that $Q^{-1}AQ$ is the diagonal matrix D. Multiplying

$$Q^{-1}AQ = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \ddots \end{pmatrix}$$

by Q on the left and Q^{-1} on the right then gives

$$A \;=\; QDQ^{-1} \tag{3.1}$$

where

$$Q = \text{ matrix whose columns are the eigenvectors (in order).} \tag{3.2}$$

Powers. From (3.1) we obtain

$$\begin{aligned} A^3 &= QDQ^{-1} \cdot QDQ^{-1} \cdot QDQ^{-1} \\ &= QD^3Q^{-1}. \end{aligned}$$

Similarly for the k^{th} power of A

$$A^k \;=\; QD^kQ^{-1} \;=\; Q \begin{pmatrix} (\lambda_1)^k & & 0 \\ & (\lambda_2)^k & \\ 0 & & \ddots \end{pmatrix} Q^{-1} \tag{3.3}$$

Polynomials. Applying (3.3) to each term in a polynomial, we have

$$A^3 + 4A^2 - 7A + 2I_n \;=\; Q\left(D^3 + 4D^2 - 7D + 2I_n\right)Q^{-1}$$

and similarly for any polynomial in A. Note that *polynomials in D are easy to calculate.*

Exponentials. For real numbers x, e^x can be defined in 3 ways:

(a) As repeated multiplication, e.g. $e^3 = e \cdot e \cdot e$.

(b) By the power series $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$.

(c) As the unique solution of the initial value problem $y'(x) = y(x),\;\; y(0) = 1$.

Each has advantages: (a) is intuitive, (b) is good for calculating, and (c) is good for showing the basic property $e^{a+b} = e^a \cdot e^b$, which is not clear from (b)! If we replace x by a square matrix A, then definition (a) makes little sense, but (b) and (c) both do, as follows.

Definition. *For a square matrix A, set*

$$e^A \;=\; I \;+\; A \;+\; \tfrac{1}{2!}A^2 \;+\; \tfrac{1}{3!}A^3 \;+\; \cdots \;+\; \tfrac{1}{n!}A^n \;+\; \cdots \tag{3.4}$$

This series converges absolutely for all matrices A.

Theorem 5. *Suppose that $A = QDQ^{-1}$ where D is diagonal. Then $e^A = Qe^DQ^{-1}$. Thus if A has a basis of eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_n$ then*

$$e^{tA} = Q\begin{pmatrix} e^{t\lambda_1} & & 0 \\ & e^{t\lambda_2} & \\ 0 & & \ddots \end{pmatrix}Q^{-1}$$

where Q is as in (3.2).

Homework III — Due Monday, Dec. 3.

If you have not yet done so, finish all problems from the previous two HW sets.

1. (a) For $A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$, find Q and D so that $A = QDQ^{-1}$.
   (b) Use this to find A^3.
   (c) Check your answer by finding A^3 directly.
2. For the matrix A in Problem 1, find e^A.
3. Suppose that B is an $n \times n$ diagonalizable matrix whose eigenvalues are all 1 or -1. Prove that $B^{-1} = B$.
4. Find e^{tC} for $C = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}$. *Solution:* $\begin{pmatrix} 3-2e^{-t} & -6+6e^{-t} \\ 1-e^{-t} & -2+3e^{-t} \end{pmatrix}$.
5. Do Problem 7 in Section 6.3: Show that, for any choice of a and b, the matrix $A = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & b \end{pmatrix}$ is *defective*, i.e. that it has fewer than 3 linearly independent eigenvectors. This means that A is not diagonalizable.

Complex Eigenvalues

Many polynomials with real coefficients have no real roots. Consequently, there are many matrices with real entries that have no real eigenvalues.

Example 1. The characteristic polynomial of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is $p_A(\lambda) = \lambda^2 + 1$, which has no real roots.

You may recognize this matrix A as a rotation of the plane counter-clockwise by 90° (notice that A takes $\mathbf{e}_1$ to $\mathbf{e}_2$ and $\mathbf{e}_2$ to $-\mathbf{e}_1$). Thus A is not a "disguised dilation" — there is no basis in which A is diagonal.

There is a simple mathematical trick that gets around this problem: regard real matrices as special cases of complex matrices, and find *complex* eigenvalues and eigenvectors. This is called "working over $\mathbb{C}$".

Example 1'. The above characteristic polynomial $p_A(\lambda) = \lambda^2 + 1$ factors over $\mathbb{C}$ as $\lambda^2 + 1 = (\lambda - i)(\lambda + i)$, so A, *regarded as a complex matrix*, has eigenvalues $\lambda = \pm i$.

In fact, eigenvalue problems can always be solved over $\mathbb{C}$. This was one of the main reasons why the definition of vector space allowed, from the beginning, scalars to be in a field F which could be $\mathbb{R}$ or $\mathbb{C}$.

Review of complex numbers. Here is a quick summary.

The *complex plane* is $\mathbb{C} = \mathbb{R}^2$ with basis $\{1, i\}$. Elements of the complex plane are called complex numbers. Thus they can be written as $\alpha = a + bi$ where $a, b \in \mathbb{R}$, and are added as vectors in $\mathbb{R}^2$:

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

Complex numbers can also be multiplied using the distributive rule and the formula $i^2 = -1$:

$$\alpha\beta \;=\; (a+bi)\cdot(c+di) \;=\; ac+adi+bci+bd\,i^2 \;=\; (ac-bd)+(ad+bc)i.$$

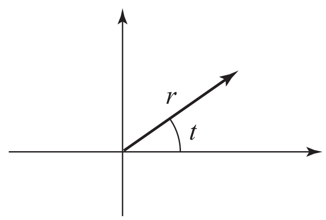
These operators are commutative, associative, and distributive. One also defines

- The *complex conjugate* of $\alpha = a+bi$ is $\overline{\alpha} = a-bi$.
- The *absolute value* or *modulus* or *norm* of α is $|\alpha| = \sqrt{a^2+b^2}$. Note that $|\alpha|^2 = \alpha\overline{\alpha}$.

Geometrically, the map $\alpha \mapsto \overline{\alpha}$ is reflection through the real axis, and $|\alpha|$ is the distance from $\alpha \in \mathbb{C}$ to the origin. One then sees that each non-zero $\alpha = a+bi \in \mathbb{C}$ has a multiplicative inverse, namely

$$\frac{1}{\alpha} \;=\; \frac{\overline{\alpha}}{\alpha\overline{\alpha}} \;=\; \frac{\overline{\alpha}}{|\alpha|^2} \;=\; \frac{a-bi}{a^2+b^2}$$

These properties mean that $\mathbb{C}$ is a *field.*



Complex numbers can also be written in polar form. As usual, points (a,b) in the plane have polar coordinates (r,θ) as shown in the figure. Then $a = r\cos\theta$ and $b = r\sin\theta$; with complex number notation this becomes

$$\alpha \;=\; a+bi \;=\; r(\cos\theta + i\sin\theta) \;=\; re^{i\theta}$$

where $r = |\alpha|$ is the modulus and θ is called the *argument of* α, and where the last equality comes from this famous fact:

Euler's formula: $e^{i\theta} \;=\; \cos\theta + i\sin\theta.$

Proof. $f(\theta) = e^{i\theta}$ is the unique solution of the initial value problem $f'(\theta) = if(\theta)$ and $f(0) = 1$. But $g(\theta) = \cos\theta + i\sin\theta$ satisfies $g'(\theta) = ig(\theta)$ and $g(0) = 1$, so $g(\theta) = e^{i\theta}$. □

All of algebra extends to the complex numbers. For example, a polynomial of degree n in a complex variable z has the form

$$p(z) \;=\; a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

where the coefficients a_i are complex numbers, i.e. $a_i \in \mathbb{C} \;\; \forall i$. A complex number $r \in \mathbb{C}$ is a *root* of $p(z)$ if $p(r) = 0$. The advantage of working over $\mathbb{C}$ is a result of the following remarkable fact.

The Fundamental Theorem of Algebra. *If $p(z)$ is a polynomial of degree $n \geq 1$ in a complex variable z, then there are n complex numbers $r_1, \ldots, r_n$ (not necessarily distinct) such that*

$$p(z) \;=\; a_n(z-r_1)(z-r_2)\cdots(z-r_n).$$

Thus over $\mathbb{C}$, every degree n polynomial has exactly n complex roots. Applying this to the characteristic polynomial of matrix, we have:

Corollary 4.4. *Every $n \times n$ matrix A over $\mathbb{C}$ has exactly n eigenvalues – the roots of $p_A(\lambda) = \det(A - \lambda I)$.*

Diagonalizing over the complex numbers. Once we agree to work over C, *the process of diagonalizing a matrix is exactly as before.* In fact, the process is often faster because of the following fact:
Lemma. *For a matrix A with real entries, the eigenvalues are of two types:*

- *real roots r_i*
- *conjugate pairs $\lambda_j, \overline{\lambda_j}$.*

and the corresponding eigenvectors are in corresponding conjugate pairs $\{\mathbf{v}_i, \overline{\mathbf{v}_i}\}$.

Proof. Suppose that $A\mathbf{v} = \lambda\mathbf{v}$, so $\mathbf{v}$ is an eigenvector with eigenvalue λ. Then since A is real,

$$A\overline{\mathbf{v}} \;=\; \overline{A\mathbf{v}} \;=\; \overline{\lambda\mathbf{v}} \;=\; \overline{\lambda}\,\overline{\mathbf{v}}$$

and thus $\overline{\mathbf{v}}$ is an eigenvector with eigenvalue $\overline{\lambda}$. Finally, note that $\lambda = \overline{\lambda} \Leftrightarrow \lambda$ is real and $\mathbf{v} = \overline{\mathbf{v}} \Leftrightarrow \mathbf{v}$ is real.
□

Example 1". The rotation matrix

$$R_\theta \;=\; \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{4.1}$$

has eigenvalues $e^{\pm i\theta}$ (a conjugate pair), and eigenvectors $\mathbf{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\overline{\mathbf{v}} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. One can then find a complex matrix Q so that

$$Q^{-1}R_\theta Q \;=\; \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Example 2. Diagonalize the matrix $A = \begin{pmatrix} 0 & 1 \\ -5 & 4 \end{pmatrix}$.

Solution. The characteristic polynomial is $p(\lambda) = -\lambda(4-\lambda) + 5 = \lambda^2 - 4\lambda + 5$. Using the quadratic formula, the roots are

$$\lambda \;=\; \frac{4 \pm \sqrt{16-20}}{2} \;=\; 2 \pm \tfrac{1}{2}\sqrt{-4} \;=\; 2 \pm i.$$

To find an eigenvector for $\lambda_1 = 2 + i$ we must solve $(A - \lambda_1 I)\mathbf{v} = 0$, or

$$\begin{pmatrix} -2-i & 1 \\ -5 & 2-i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note that the bottom row is $2 - i$ times the top row because $(2-i)(-2-i) = -5$. This leaves only the top row, which gives the relation $-(2+i)a + b = 0$. For one solution, take $a = 1$ to get the eigenvector

$$\mathbf{v}_1 \;=\; \begin{pmatrix} 1 \\ 2+i \end{pmatrix}$$

We get the second eigenvector by taking the conjugate:

$$\mathbf{v}_2 \;=\; \overline{\mathbf{v}}_1 \;=\; \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

The matrix is now diagonalized by $Q = \begin{pmatrix} 1 & 1 \\ 2+i & 2-i \end{pmatrix}$ which has $\det Q = (2-i) - (2-i) = -2i$. Then

$$Q^{-1}AQ \;=\; \frac{1}{-2i}\begin{pmatrix} 2-i & -1 \\ -2-i & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2+i & 2-i \end{pmatrix} \;=\; \begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}$$

Homework IV — Due Wednesday, Dec. 5.

1. (a) What is $\frac{1}{i}$? *Hint:* $i\,\alpha = 1$ for what α?

   (b) Write the number $4 - 4i$ in polar form.

2. A complex number z is called a n^{th} *root of unity* if $z^n = 1$.

   (a) How many n^{th} roots of unity are there? (Apply the Fundamental Theorem of Algebra to $p(z) = z^n - 1$).

   (b) Write the n^{th} roots of unity $\{z_k\}$ (all of them) in polar form, i.e. $z_k = re^{i\theta}$ for what r and θ?

   (c) Draw a picture of the complex plane showing the unit circle and all of the 8^{th} roots of unity.

3. (a) If $z = re^{i\theta}$ and $w = se^{i\phi}$, what is the polar form of the product zw?

   (b) For a fixed complex number $z = re^{i\theta}$, show that there are exactly two complex numbers w with $w^2 = z$ and find the polar expressions of both of these numbers.

4. Find all eigenvalues of the following matrices.

$$A = \begin{pmatrix} 11 & -15 \\ 6 & -7 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

5. For each of the following matrices find

   (a) All complex eigenvalues.

   (b) Corresponding (complex) eigenvectors.

   (c) Matrices Q so that $Q^{-1}CQ$ and $Q^{-1}DQ$ are diagonal.

$$C = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{pmatrix}$$