## Section 4.2 Linear transformations, their matrices and their geometry

These notes cover and extend Section 4.2 in the textbook. The purpose is to illustrate two principles:

**Key principles:** for a linear transformation $L : V \to W$

- $L$ is determined by what it does to the basis elements.
- After we choose a basis for $V$ and a basis for $W$, $L$ is described by a matrix $A$.

The matrix of a linear transformation $L : V \to W$ is built as follows:

1. Fix a basis $C = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of $V$ and a basis $D = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ of $W$.
2. For each $v_i$ in the $C$-basis, write $L(\mathbf{v}_i)$ as a linear combination $\sum a_{ij}\mathbf{w}_j$ of the $D$-basis vectors, and put the coordinates into a column vector: $[L\mathbf{v}_i]_D = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}$
3. Build the matrix $A$ whose columns are $[L\mathbf{v}_1]_D, [L\mathbf{v}_2]_D$, etc. :
$$A \;=\; [L]_{DC} \;=\; \begin{pmatrix} | & | & \cdots & | \\ L\mathbf{v}_1 & L\mathbf{v}_2 & \cdots & L\mathbf{v}_n \\ | & | & \cdots & | \end{pmatrix}$$
This is called the *matrix of L with respect to the bases C and D*.

Then $L$ is then given by multiplication by the matrix $A$ in the following sense:

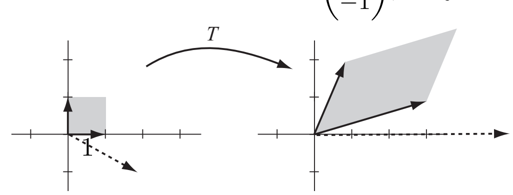
**Theorem 4.2.2** For a vector $\mathbf{x} \in V$ with coordinates $[\mathbf{x}]_C$ in the basis $C$, the $D$-coordinates of $L\mathbf{x}$ are given by
$$[L\mathbf{x}]_D \;=\; A[\mathbf{x}]_C$$
where $A = [L]_{DC}$ is the above matrix.

**Most Useful Case.** For linear transformations $L : \mathbb{R}^n \to \mathbb{R}^m$, we can use the standard bases $\{\mathbf{e}_i\}$. The procedure for finding the matrix is then easy:

1. Write $L(\mathbf{e}_1)$, $L(\mathbf{e}_2)$, $\dots$ as a column vectors.
2. Assemble these column vectors into a matrix $A$.

Then $L(\mathbf{x}) = A\mathbf{x}$, i.e. the transformation is given by matrix multiplication by $A$.

**Example 1.** Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that takes $\mathbf{e}_1 = (1, 0)^T$ to $(3, 1)^T$ and $\mathbf{e}_2 = (0, 1)^T$ to $(1, 2)^T$. In column form
$$L\begin{pmatrix}1\\0\end{pmatrix} \;=\; \begin{pmatrix}3\\1\end{pmatrix} \qquad \text{and} \qquad L\begin{pmatrix}0\\1\end{pmatrix} \;=\; \begin{pmatrix}1\\2\end{pmatrix},$$
so the matrix of $L$ is $A = \begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}$. To find where $L$ takes the vector $\begin{pmatrix}2\\-1\end{pmatrix}$, we just multiply by $A$:
$$L\begin{pmatrix}2\\-1\end{pmatrix} \;=\; \begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}5\\0\end{pmatrix}.$$



**Example 2.** What is the matrix of the linear transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x - 2y \\ -x + 7y \end{pmatrix}$?

Solution: taking $x = 1$ and $y = 0$, and then $x = 0$ and $y = 1$, we obtain:

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}, \qquad \text{so the matrix is} \quad A = \begin{pmatrix} 4 & -2 \\ -1 & 7 \end{pmatrix}.$$

**Examples of 6 types of linear transformations $L : \mathbb{R}^2 \to \mathbb{R}^2$.**

1. *Dilation* by factor of 3 horizontally, factor of 5 vertically: $D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$.
2. *Rotation* counterclockwise by 90°: $R_{90} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and by angle $\theta$: $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
3. The *reflection* across the $x$-axis: $R_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
4. A *shear* that fixes the first basis vector $\mathbf{e}_1$ and moves $\mathbf{e}_2$ to $\mathbf{e}_1 + \mathbf{e}_2$: $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
5. The *orthogonal projection* onto the $x$-axis: $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
6. The *embedding* $E : \mathbb{R}^2 \to \mathbb{R}^3$ that takes $\mathbb{R}^2$ into the $xy$-plane: $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

## Supplemental Homework Problems

1. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that takes $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $3\mathbf{e}_1 + 4\mathbf{e}_2$ and takes $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $-2\mathbf{e}_1 + 5\mathbf{e}_2$. What is the matrix for $L$ with respect to the (standard) basis $\{\mathbf{e}_1, \mathbf{e}_2\}$?
2. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation about the origin by 60°.
   (a) What is $R(\mathbf{e}_1)$? What is $R(\mathbf{e}_2)$?
   (b) What is the matrix of $R$ with respect to the standard basis?
   (c) What is the image $R(\mathbf{v})$ of the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?
3. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the $yz$ plane, given by $P(x, y, z) = (0, y, z)$.
   (a) What are $P(\mathbf{e}_1)$, $P(\mathbf{e}_2)$ and $P(\mathbf{e}_3)$?
   (b) What is the matrix of $L$ with respect to the standard basis?
   (c) What is the image $P(\mathbf{v})$ of the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$?
4. (a) Using the formula given in Example 2 above, find the matrix of the rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ about the origin by 30°.
   (b) This rotation takes $\begin{pmatrix} 4 \\ 9 \end{pmatrix}$ to what vector?

5. Given points $\mathbf{v}_0$ and $\mathbf{v}_1$ in a vector space, the line segment between them is

$$S = \left\{ v_t = (1-t)\mathbf{v}_0 + t\mathbf{v}_1 \;\middle|\; 0 \le t \le 1 \right\}.$$

Prove that a linear transformation $L : V \to W$ takes each line segment to the line segment between the image points $L(\mathbf{v}_0)$ and $L(\mathbf{v}_1)$.

6. Let $\square$ denote the unit square in $\mathbb{R}^2$ with corners at $(0,0), (1,0), (0,1)$ and $(1,1)$. Using the fact that you proved in Problem 4, sketch the image of $\square$ under the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix is

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

7. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the $z$-axis. What are $P(\mathbf{e}_1), P(\mathbf{e}_2)$ and $P(\mathbf{e}_3)$? What is the matrix for $P$?

8. The "anti-diagonal" line $L^-$ in $\mathbb{R}^2$ is the graph of $y = -x$, which can also be defined as

$$L^- = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}.$$

Reflection through $L^-$ (obtained by moving along the line segment perpendicular to $L^-$, going an equal distance to the opposite side of $L^-$) is a linear transformation, which we denote by $R$.

   (a) What is $R\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? What is $R\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

   (b) Find the matrix of this reflection $R$.

   (c) Where does $R$ take the vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$?

9. Find a non-zero $2 \times 2$ matrix $A$ such that $A\mathbf{v}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for all $\mathbf{v} \in \mathbb{R}^2$. *Hint: Find a vector* $\mathbf{w}$ *perpendicular to* $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *and define* $A$ *by* $A\mathbf{e}_1 = \mathbf{w}$ *and* $A\mathbf{e}_2 = r\mathbf{w}$ *for your favorite* $r \in \mathbb{R}$.

10. Find the matrices of the following transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$.

   (a) The reflection across the $xz$-plane.

   (b) The rotation about the $z$-axis through an angle $\theta$ counterclockwise as viewed from the positive $z$-axis.

   (c) Reflection across the plane $y = z$.

11. Let $D$ be the dilation $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$. Show that $D$ takes the unit circle $x^2 + y^2 = 1$ to an ellipse. Sketch the ellipse. *Hint: What is the image point* $\begin{pmatrix} z \\ w \end{pmatrix}$ *of* $\begin{pmatrix} x \\ y \end{pmatrix}$*? What equation do* $z$ *and* $w$ *satisfy?*

12. Let $T : P_3 \to P_3$ be the linear transformation that takes a polynomial $p(x) = ax^3 + bx^2 + cx + d$ to $p(x-2)$ (i.e. replace $x$ by $x - 2$).

    What is the matrix for $T$ with respect to the basis $\{x^3, x^2, x, 1\}$ of $P_3$? (Be sure to keep the basis elements in this order.)

## Solutions to Selected Problems on the "Section 4.2 Handout"

1. The matrix for $L$ is $\begin{pmatrix} | & | \\ L\mathbf{v}_1 & L\mathbf{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 4 & 5 \end{pmatrix}$.

2. Using the definition of sin and cos, $R(\mathbf{e}_1) = \begin{pmatrix} \cos 60° \\ \sin 60° \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$. Similarly, $R(\mathbf{e}_2) = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$. Hence

$$R = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad R\mathbf{v} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1-\sqrt{3} \\ 1+\sqrt{3} \end{pmatrix}.$$

3. Taking $x = 1$ and $y = z = 0$ in the formula for $P$ gives $P(\mathbf{e}_1) = 0$. Similarly, $P(\mathbf{e}_2) = \mathbf{e}_2$ and $P(\mathbf{e}_3) = \mathbf{e}_3$. Hence

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

4. As in Problem 2, one finds that

$$R = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \quad \text{and} \quad R\mathbf{v} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 4 \\ 9 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4\sqrt{3}-9 \\ 4+9\sqrt{3} \end{pmatrix}.$$

5. For each $t$, $0 \le t \le 1$, we can take $\alpha = 1-t$ and $\beta = t$ in the definition of linear map to find that $L((1-t)\mathbf{v}_0 + t\mathbf{v}_1) = (1-t)L\mathbf{v}_0 + tL\mathbf{v}_1$. The righthand side is the line segment from $L\mathbf{v}_0$ to $L\mathbf{v}_1$.

6. Matrix multiplication shows that $L(0,0)^T = (0,0)^T$, $L(1,0)^T = (1,0)^T$, $L(0,1)^T = (-1,1)^T$ and $L(1,1)^T = (0,1)^T$. Plot these points in $\mathbb{R}^2$ and joint with line segments gives a parallelogram in $\mathbb{R}^2$ that is the image $L(\square)$.

7. $P(\mathbf{e}_1), P(\mathbf{e}_2)$ and $P(\mathbf{e}_3)$ are the three columns of the matrix of $P$, which is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8. Done in class.

9. Done in class.

10. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. (b) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

    (c) $\mathbf{e}-1$ is in this plane, so $R(\mathbf{e}_1) = \mathbf{e}_1$, while $R(\mathbf{e}_2) = \mathbf{e}_3$ and $R(\mathbf{e}_3) = \mathbf{e}_2$. Thus $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

11. If $x^2 + y^2 = 1$ then $\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 5y \end{pmatrix}$ satisfies $\left(\frac{z}{3}\right)^2 + \left(\frac{w}{5}\right)^2 = 1$. This is the equation of the ellipse that intersects the $x$-axis at $(\pm 3, 0)$ and intersects the $y$-axis at $(0, \pm 5)$.

12. Applying $T$ to each of the basis vectors $\{x^3, x^2, x, 1\}$ (in this order!), gives

    - $T(x^3) = (x-2)^3 = x^3 - 6x^2 + 12x - 8$
    - $T(x^2) = (x-2)^2 = x^2 - 8x + 4$
    - $T(x) = (x-2) = x - 2$
    - $T(1) = (1-2) = -1$

    so the matrix for $T$ is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 \\ 12 & -8 & 1 & 0 \\ -8 & 4 & -2 & -1 \end{pmatrix}$.