1. (20 pts, 4 pts each) State clearly and concisely the following definitions.

   (a) An $n \times n$ matrix A is **nonsingular**.

   (b) A set S is a **subspace** of a vector space V.

   (c) Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a vector space V are **linearly independent**.

   (d) A set of vectors $\mathbf{x}$ in $\mathbb{R}^n$ is the **orthogonal complement** of a subspace Y of $\mathbb{R}^n$

   (e) An operation $\langle\ ,\ \rangle$ is an **inner product** on a vector space V.

2. (15 pts) Let a and b be real numbers and consider the following linear system:

$$\begin{array}{rcrcrcl} x_1 & + & 5x_2 & + & 2x_3 & = & b \\ & & x_2 & + & 3x_3 & = & b^2 \\ & & 2x_2 & + & ax_3 & = & 4 \end{array}$$

(a) Find all values of a and b such that the system has no solutions.

(b) Find all values of a and b such that the system has a unique solution.

(c) Find all values of a and b such that the system has infinitely many solutions.

3. (15 pts, 5 pts each) Determine whether the set S given below is a subspace of a vector space V.

   (a) S is a set of all **skew-symmetric** $n \times n$ matrices A (that is $A^{\mathrm{T}} = -A$) in the set V of all $n \times n$ matrices.

   (b) S is a set of all **orthogonal** $n \times n$ matrices Q (that is $Q^{\mathrm{T}} = Q^{-1}$) in the set V of all $n \times n$ matrices.

   (c) S is a set of all continuous functions $f(x) \in C[0,1] = V$ such that $f(0) \geq 0$.

4. (15 pts) Let $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -2 & 3 & 0 \\ -3 & 3 & -4 & 1 \\ 3 & -3 & 4 & -1 \end{bmatrix}$. Use row operations to find the basis in the **row-vector** space $R[A^{\mathrm{T}}]$, **column-vector** space $R[A]$, and the basis in $N[A]$. What is the interrelation between $R[A^{\mathrm{T}}]$ and $N[A]$?

5. (15pts) Let $L : P_2 \to P_3$ be a mapping defined by

$$L(p(x)) = 2p(x) + x^2 p'(x).$$

(a) verify that L is a **linear transformation**

Find the matrix representation of L with respect to the ordered bases $[1 + x, 1 - x]$ of P_2 and $[x^2, x, 1]$ of P_3.

6. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$.

(a) (6pts) Find all eigenvalues of A

(b) (8pts) for each of the eigenvalue find the corresponding eigenvectors.

(c) (6pts) Find a nonsingular matrix X and a diagonal matrix D such that $A = XDX^{-1}$.

7. (15pts) Let $L : V \to W$ be a linear transformation from a vector space V to a vector space W and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in V. Prove that if the vectors $\mathbf{w}_1 = L(\mathbf{v}_1)$, $\mathbf{w}_2 = L(\mathbf{v}_2)$, $\ldots$, $\mathbf{w}_n = L(\mathbf{v}_n)$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ must be linearly independent.

8. (15pts) Consider the vector space $\mathbb{R}^n$ with the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{y}$ and an $n \times n$ matrix A.

   (a) Prove that for any $\mathbf{x}$ in $\mathbb{R}^n$, $\langle \mathbf{x}, A^{\mathrm{T}}A\mathbf{x} \rangle = \|a\mathbf{x}\|^2$.

   (b) Prove that if λ is an eigenvalue of the matrix $A^{\mathrm{T}}A$, then $\lambda \geq 0$.

9. (a) (6pts) Let λ be a real number. Prove by induction that for any $n \geq 1$,

$$A^n = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

(b) (6pts) Using your answer to part (a) to find e^A for the matrix $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Use that $e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

10. (a) (6pts) Find the projection of the function x on the function $\sin(kx)$ in the space $C[-\pi, \pi]$ with the inner product $\langle p, q\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} p(x)q(x)dx$

(b) (6pts) Find the dimension of the null space of A^{T} if A is a 7×9 matrix whose rank is 4.

(c) Let A be a defective 3×3 matrix with two distinct eigenvalues λ_1 and λ_2. Find $\dim N[A - \lambda_1 I]$ and $\dim N[A - \lambda_2 I]$