Fall 2018
Prof. T Parker

Course Review: Math 309

To study for the Math 309 Final:

- Read through your class notes and, as you go, *read the Homework problems* and be sure that you can now solve them.
- Do the same for all of the handouts that are posted on the class webpage.
- Look over the Review Sheets for Exams 1 and 2, your own Exams 1 and 2, and the Exam solutions handed out in class.
- Do the Review Problems below.

Linear Systems and Determinants

1. Solve systems of linear equations $A\mathbf{x} = \mathbf{b}$: Gaussian elimination, elementary row operations, reduced row-echelon form.
2. Know the basic theorems on linear systems (e.g. the Theorem on page 20 of the textbook).
3. The definition of the inverse A^{-1} of a square matrix, how to find it using row operations, and solving $A\mathbf{x} = \mathbf{b}$ by $\mathbf{x} = A^{-1}\mathbf{b}$.
4. Finding the rank and nullity of a matrix and the Rank-Nullity Theorem.
5. Compute determinants (i) by cofactor exansions, and (ii) by row reduction.
6. Know how to use the basic facts about determinants: $\det AB = \det A \det B$, and A is invertible $\Leftrightarrow \det A \neq 0$.
7. Know the equivalent ways of characterizing "non-singular": A is non-singular $\Leftrightarrow \det A \neq 0 \Leftrightarrow\ \exists$ a unique solution to $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}\ \Leftrightarrow \cdots$.

Vector spaces

1. Vector spaces: Be able to proof simple consequences of the axioms. Know the four main examples $V = \mathbb{R}^n,\ \mathbb{R}^{m\times n},\ \mathbf{P}_n,\ \mathcal{C}[a,b]$.
2. Definition of a subspace. Proof that the intersection of two subspaces is a subspace.
3. Linear combinations and the span; how to determine if a vector is in the span.
4. Linear independence/dependence. How to determine if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are linearly independent or not.
5. Basis. How to check if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis. The Two-for-One Theorem.
6. Definition of dimension. The dimensions of the standard vector spaces $\mathbb{R}^n$, $P_n(\mathbb{R})$ and $M_{\mathbb{R}}(n, m)$.
7. How to obtain bases using the Expanding and Paring Down Theorems.
8. Given a matrix A, find a basis for the null space $N(A)$ and the column space $R(A)$.
9. Coordinate vectors: how to find the coordinates of a given vector by solving a linear system.

Linear transformations

1. Linear transformations $L : V \to W$: the definition, examples and basic properties.

2. The definitions of the nullspace $N(L)$ and the range $R(L)$, the proofs that these are subspaces, and the definitions of the rank and nullity of L.

3. Know how to determine the nullspace and range for $L_A : \mathbb{R}^n \to \mathbb{R}^m$ given as a matrix A (using row reduction and pivot columns).

4. How to find the matrix $[L]_{FE}$ of a linear transformation: $[L\mathbf{v}]_F = [L]_{FE}\,[\mathbf{v}]_E$

5. How to write down matrices for rotations and dilations of $\mathbb{R}^2$ and $\mathbb{R}^2$: see the problems in the "Section 4.2" handout (on class webpage).

6. Change bases: $[L]_{FF} = Q^{-1}\,[L]_{EE}\,Q$ where $Q =$ the matrix whose columns are the coordinates of the vectors of $F = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in the E-basis.

Inner product spaces

1. Compute dot products, norms, the component of $\mathbf{v}$ along $\mathbf{w}$, and the projection $Proj_{\mathbf{v}}(\mathbf{w})$.

2. Check if subspaces are orthogonal. Given a subspace S, find a basis of $S^{\perp}$.

3. Solve Least Squares Problems: the LS solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b}$. Understand the geometric meaning of the LS solution in terms of orthogonal projections.

4. Given an inner product space, apply the Gram-Schmidt process to a list of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$.

5. Given vectors $\mathbf{v} = \sum a_i\mathbf{e}_i$ and $\mathbf{w} = \sum b_i\mathbf{e}_i$ expanded in an orthonormal basis $\{\mathbf{e}_i\}$, find $\langle \mathbf{v}, \mathbf{w}\rangle$, and $\mathrm{Proj}_{\mathbf{v}}(\mathbf{w})$.

6. If $\{\mathbf{e}_i\}$ is an orthonormal basis of a subspace S, find the projection $\mathrm{Proj}_S(\mathbf{v}) = \sum \langle \mathbf{v}, \mathbf{e}_i\rangle\,\mathbf{e}_i$.

Eigenvalues and Diagonalizability

1. The definitions of *eigenvalue, eigenvector, characteristic polynomial* $p_A(\lambda)$, and *eigenspace* $E_A(\lambda)$.

2. How to find eigenvalues and the corresponding eigenvectors.

3. How to diagonalize a matrix.

4. Be able to prove that (a) Each eigenspace $E_A(\lambda)$ is a subspace, (b) eigenvectors with distinct eigenvalues are linearly independent, and (c) linear transformations with distinct eigenvalues are diagonalizable over $\mathbb{C}$.

5. The definition of *similar matrices*; these have same trace, same determinant, same eigenvalues.

6. Given a matrix A, use diagonalization to find A^k and e^A.

Miscellaneous Review Problems (mostly from Chapters 5 and 6)

1. Solve the linear system 5(j) on page 24 of the textbook. Write the solution set in set notation.

2. Use row reduction to find the inverse of the matrix $A = \begin{pmatrix} 1 & 5 \\ -2 & -6 \end{pmatrix}$.

3. (a) Are the vectors $\mathbf{v}_1 = (1, 2, 4)^T$, $\mathbf{v}_2 = (2, 1, 3)^T$, $\mathbf{v}_3 = (4, -1, 1)^T$ in $\mathbb{R}^3$ linearly independent?
   (b) Are the polynomials $p = x^3 - 2x$, $q = x^2 + 4x + 1$, $r = x^3 - x^2 + 1$ in P_4 linearly independent?

4. Find bases for $N(A)$ and $R(A)$ for the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}$.

5. Prove that the set of all $n \times n$ matrices A with $\operatorname{tr} A = 0$ is a vector subspace of $\mathbb{R}^{n\times n}$.

6. Complete the following definitions and memorize the full definition. *Be precise! Don't just make up a definition, find the definition in the book, your notes, or a handout.*

   (a) An *inner product* on a vector space V associates to each pair of vectors $\mathbf{v}, \mathbf{w} \in V$ a number $\langle \mathbf{v}, \mathbf{w} \rangle \in \mathbb{R}$ that satisfies ...

   (b) In an inner product space V, vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are:
   - an orthonormal set if ...
   - an orthonormal basis if ...

   (c) The *orthogonal projection* $\operatorname{Proj}_{\mathbf{w}}(\mathbf{v})$ of $\mathbf{v}$ onto $\mathbf{w}$ is given by the formula ...

7. In $\mathbb{R}^4$ with the dot product, consider the vectors $\mathbf{v} = (2, 3, -1, 4)$ and $\mathbf{w} = (5, 2, -3, 8)$. Find $\langle \mathbf{v}, \mathbf{w} \rangle$, $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, and $\operatorname{Proj}_{\mathbf{w}}(\mathbf{v})$.

8. Consider the inner product space $V = \mathbb{R}^{2\times 3}$ with $\langle A, B \rangle = \operatorname{tr}(B^T A)$. Find $\cos\theta$, where θ is the angle between

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 2 \\ 5 & 3 \\ 0 & 1 \end{pmatrix}.$$

   Trick: To find $\operatorname{tr}(B^T A)$ you need only find the diagonal entries of the square matrix $B^T A$ — skip the others!

9. (a) Find the orthogonal projection of $\mathbf{v} = (2, 0, -1, 3)$ onto $\mathbf{w} = (1, 1, 0, 1)$.

   (b) In $\mathbb{R}^4$, write $\mathbf{v}$ as the sum of a constant times $\mathbf{w}$ and a vector $\mathbf{x}$ orthogonal to $\mathbf{w}$.

   (c) Draw a picture of your solution in the plane spanned by $\mathbf{v}$ and $\mathbf{w}$.

10. Consider the functions $f(x) = x^2$ and $g(x) = 3x - 1$ in $\mathcal{C}[-1, 1]$ with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$.

    (a) Find the orthogonal projection of f onto g.

    (b) Write f as the sum of a constant times g and a function h orthogonal to g.

11. Suppose that $\{\mathbf{e}_i\}$ is an orthonormal basis of a real inner product space V. Prove that if $\mathbf{v} = \sum_i a_i\, \mathbf{e}_i$ and $\mathbf{w} = \sum_j b_j\, \mathbf{e}_j$ then

$$a_j = \langle \mathbf{v}, \mathbf{e}_j \rangle \qquad \text{and} \qquad \langle \mathbf{v}, \mathbf{w} \rangle = \sum_i a_i b_i.$$

12. Let W be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{e}_1 = \frac{1}{\sqrt{2}}(1, 0, 1, 0)$ and $\mathbf{e}_2 = \frac{1}{\sqrt{6}}(-1, 2, 1, 0)$.

    (a) Verify that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal set in $\mathbb{R}^4$.

    (b) What is the dimension of W?

    (c) Find the projection of $\mathbf{v} = (0, 1, 2, 3)$ onto W.

    (d) What is the distance from $\mathbf{v}$ to W?

    (e) Apply the Gram-Schmidt process to modify $\mathbf{v}$ to obtain a vector $\mathbf{e}_3$ so that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal set in $\mathbb{R}^4$.

13. The matrix $A = \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$ defines a linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$. Use the Gram-Schmidt process to find an orthonormal basis for the range $R(A)$.

    Hint: Find the image vectors $A\mathbf{e}_1$ and $A\mathbf{e}_2$ of the standard basis vectors, and apply Gram-Schmidt.

14. For $V = \mathcal{C}[-\pi,\pi]$ with the inner product $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$, the textbook shows that the functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$ are an orthonormal set. Let S be the space of these functions. What is the orthogonal projection of $f(x) = x^2$ onto S? Express the coefficients as integrals, but *do not do the integrals.*

15. (a) Let A be the matrix of a linear transformation $\mathbb{R}^k \to \mathbb{R}^n$ with $n > k$. Write down the general formula for the Least Squares solution to the equation $A\mathbf{x} = \mathbf{b}$.
    (b) Draw and label a picture showing the image of A, $\mathbf{x}$, and $\mathbf{b}$.

16. Find the Least Squares solution for the following system. Your answer will involve fractions.

$$\begin{aligned} x + y &= 3 \\ -2x + 3y &= 1 \\ 2x - y &= 2 \end{aligned}$$

17. Let $T : V \to V$ be a linear operator. Prove that the eigenspace $E_T(\lambda)$ is a vector subspace of V.

18. True or False?
    (a) If an $n \times n$ matrix has n distinct eigenvectors, then A is diagonalizable.
    (b) If A is diagonalizable, so is A^T.
    (c) If A is a diagonalizable $n \times n$ matrix, then there exist eigenvectors of A that form a basis for $\mathbb{R}^n$.

19. Let A be an invertible $n \times n$ matrix. Prove that if λ is an eigenvalue of A with eigenvector $\mathbf{u}$, then λ^{-1} is an eigenvalue of A^{-1} with eigenvector $\mathbf{u}$.

20. Find the eigenvalues and eigenvectors for the matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -3 & -1 \\ -1 & 2 & 1 \\ 3 & -9 & -4 \end{pmatrix}.$$

21. Diagonalize the matrices

$$G = \begin{pmatrix} 2 & 6 \\ 1 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Specifically, find all eigenvalues, find the corresponding eigenvectors, and find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

22. For the matrix G above, find G^5 and e^{tG} find G^5 and e^{tG} (use your answer to Problem 20).

23. Do this problem using the TRICK that $\operatorname{tr}(A) =$ sum of e-values, and $\det A =$ product of e-values.

    (a) Find the eigenvalues for the following matrices (no need to find the eigenvectors).

$$(a)\ \begin{pmatrix} 5 & 11 \\ 1 & -5 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}$$

    (b) Consider the matrix $A = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -1 & 5 \\ -1 & 1 & 0 \end{pmatrix}$.

        (i) Check that $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ is an eigenvector. What is its eigenvalue?

        (ii) Use the trick to find the other two eigenvectors of A.