Definitions

For a function of two variables $f(x,y)$, and a unit vector $\vec{u} = \langle u_1, u_2 \rangle$:

- The **directional derivative** of f at (a,b) in the direction $\vec{u}$ is

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(a + u_1 h, b + u_2 h) - f(a,b)}{h}$$

- The **gradient** of f is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

  Specifically, at the point (a,b):

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$$

  If $f(x,y,z)$ is a function of three variables, then

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Important Formulas

- The directional derivative of f in the direction of the unit vector $\vec{u}$ is given by the formula

$$D_{\vec{u}} f(a,b) = \vec{u} \cdot \nabla f(a,b)$$

- The chain rule can be re-phrased in terms of the gradient vector. If $\vec{r}(t) = \langle x(t), y(t) \rangle$, and $f = f(x(t), y(t))$, then

$$\frac{df}{dt} = \vec{r}\,'(t) \cdot \nabla f(x(t), y(t))$$

Maximum Rate of Change

- The function f increases the fastest in the direction of ∇f, with the maximum rate of increase being $|\nabla f|$.
- Likewise, the function f decreases the fastest in the direction of $-\nabla f$.

Level Sets

- If $f(x,y)$ is a function of two variables, then the level sets $f(x,y) = c$ are curves in $\mathbb{R}^2$. Then at a point on the curve, ∇f is orthogonal to the curve (meaning orthogonal to the tangent line).
- If $f(x,y,z)$ is a function of three variables, then the level sets $f(x,y,z) = c$ are surfaces in $\mathbb{R}^3$. Then at a point on the surface, ∇f is orthogonal to the surface (meaning orthogonal to the tangent plane).