Tangent Planes

For a function of two variables $f(x, y)$:

- The tangent plane to the graph $z = f(x, y)$ at the point (a, b) has normal vector

$$\vec{n} = \langle f_x(a, b), f_y(a, b), -1\rangle$$

  This is obtained as the cross product

$$\langle 0, 1, f_y(a, b)\rangle \times \langle 1, 0, f_x(a, b)\rangle$$

- The equation for the tangent plane is then

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$$

or

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Linearization

- The **linearization** of $f(x, y)$ is the linear function whose graph is the tangent plane to the graph of f. That is, the linearization at (a, b) is the function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- You can approximate $f(x, y)$ when (x, y) is close to (a, b) using L instead of the original f.

Differentials

- The **differential** of f is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

The Chain Rule

- If $z = f(x, y)$, and $x = x(t)$ and $y = y(t)$, then

$$z'(t) = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

or

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

- If $z = f(x, y)$, and x and y depend on two variables s and t (that is, $x = x(t, s)$ and $y = y(t, s)$), then

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

and

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$