Surface Integrals of Scalar Functions

Let $f(x, y, z)$ be a scalar function of three variables, and S a surface given parametrically by

$$\vec{r}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$$

where (u, v) are in some parameter domain D (in the u, v plane). Then the surface integral of f over S is defined to be

$$\iint_S f(x, y, z) dS = \iint_D f(x(u, v), y(u, v), z(u, v))\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

In the special case that S is the graph of some function $z = g(x, y)$, then we can parameterize as

$$\vec{r}(x, y) = \langle x, y, g(x, y)\rangle$$

In this case, $\vec{r}_x = \langle 1, 0, f_x\rangle$ and $\vec{r}_y = \langle 0, 1, f_y\rangle$, giving $\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1\rangle$, and so $|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + f_x^2 + f_y^2}$. The surface integral in this case takes the special form

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\, \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

Surface Integrals of Vector Fields

Let $\vec{F}(x, y, z) = \langle P(x, y, z),\ Q(x, y, z),\ R(x, y, z)\rangle$ be a vector field, and S a parametric surface:

$$\vec{r}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$$

where (u, v) are in some parameter domain D (in the u, v plane). Also let $\vec{n}$ be the unit normal vector to the surface. Then the surface integral of $\vec{F}$ over S is defined to be

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

As above, if S is the graph of a function $z = f(x, y)$, then this integral takes a special form:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(R - \frac{\partial f}{\partial x} P - \frac{\partial f}{\partial y} Q\right)\, dx\, dy$$