Vector Fields

- A **vector field** is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ or from $\mathbb{R}^3$ to $\mathbb{R}^3$. The notation is

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$$

or

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

Conservative Fields

- A vector field $\vec{F}$ is **conservative** if there is a function f so that $\vec{F} = \nabla f$.
- If $\vec{F} = \langle P, Q\rangle$ is defined on a simply-connected region, then $\vec{F}$ is conservative if and only if

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

- The **fundamental theorem for line integrals** says that

$$\int\limits_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Line Integrals

For all of the following, assume C is a curve parameterized by $\vec{r}(t) = \langle x(t), y(t)\rangle$ with $a \le t \le b$.

- The **line integral of a function** $f(x,y)$ over C is given by

$$\int\limits_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

- The **line integral of a vector field** $\vec{F}(x,y)$ over C is given by

$$\int\limits_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)dt$$

If $\vec{F} = \langle P, Q\rangle$, then we often write $\int\limits_C P dx + Q dy$ instead of $\int\limits_C \vec{F}\cdot d\vec{r}$. In this notation, the above formula looks like:

$$\int\limits_C (Pdx + Qdy) = \int_a^b \langle P, Q\rangle \cdot \langle x'(t), y'(t)\rangle\,dt$$

- If $\vec{F}$ represents a force field, then the line integral $\int\limits_C \vec{F}\cdot d\vec{r}$ represents the work done by the force.

Green's Theorem

In the following, $\vec{F} = \langle P, Q\rangle$ is a vector field, C is a closed curve (the endpoints are the same), and D is the region inside the curve C.

- **Green's Theorem** says that

$$\oint\limits_C \vec{F}\cdot d\vec{r} = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

- If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then the line integral $\oint\limits_C \vec{F}\cdot d\vec{r}$ computes the area of D. Examples of vector fields with this property are:
  1. $\vec{F} = \frac{1}{2}\langle -y, x\rangle$
  2. $\vec{F} = \langle 0, x\rangle$
  3. $\vec{F} = \langle -y, 0\rangle$