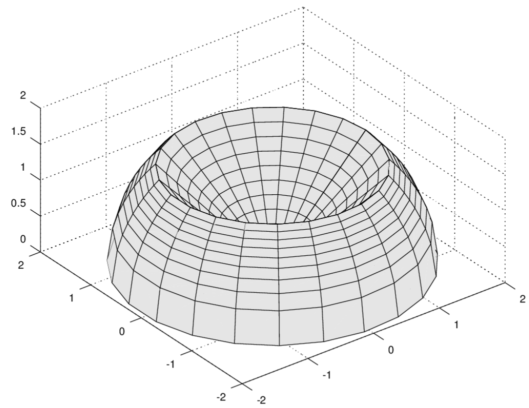
Name: ____

1. Consider the pictured region, which is inside the sphere $x^2 + y^2 + z^2 = 4$, above the x, y-plane, and under the cone $z = \sqrt{x^2 + y^2}$.



(a) (2 points) Compute the volume of this region using a triple integral in spherical coordinates.

Solution:

$$\iiint_E dV = \int_{\pi/4}^{\pi/2} \int_0^2 \int_0^{2\pi} \rho^2 \sin(\phi)\, d\theta\, d\rho\, d\phi$$
$$= 2\pi \int_{\pi/4}^{\pi/2} \int_0^2 \rho^2 \sin(\phi)\, d\rho\, d\phi$$
$$= \frac{16\pi}{3} \int_{\pi/4}^{\pi/2} \sin(\phi)\, d\phi$$
$$= -\frac{16\pi}{3} \left[\cos(\phi)\right]_{\pi/4}^{\pi/2}$$
$$= \frac{16\pi}{3} \left[\cos(\phi)\right]_{\pi/2}^{\pi/4}$$
$$= \frac{16\pi}{3\sqrt{2}}$$

Continue on to back side

(b) (2 points) Compute the volume of the same region from part (a), but using a triple integral in cylindrical coordinates.
(Hint: You will need to split it into two integrals)

Solution: In cylindrical coordinates, the equation for the sphere is $z = \sqrt{4-r^2}$, and the equation for the cone is $z = r$. So the two surfaces meet when $r = \sqrt{4-r^2}$. Solve this to get $r = \sqrt{2}$. So when $0 \le r \le \sqrt{2}$, the top of the region is the cone, and when $\sqrt{2} \le r \le 2$, the top of the region is the sphere. So we split into two integrals:

$$\begin{aligned}
\iiint_E dV &= \int_0^{\sqrt{2}} \int_0^{2\pi} \int_0^r r\,dz\,d\theta\,dr + \int_{\sqrt{2}}^2 \int_0^{2\pi} \int_0^{\sqrt{4-r^2}} r\,dz\,d\theta\,dr \\
&= \int_0^{\sqrt{2}} \int_0^{2\pi} r^2\,d\theta\,dr + \int_{\sqrt{2}}^2 \int_0^{2\pi} r\sqrt{4-r^2}\,d\theta\,dr \\
&= 2\pi \left(\int_0^{\sqrt{2}} r^2\,dr + \int_{\sqrt{2}}^2 r\sqrt{4-r^2}\,dr \right) \\
&= 2\pi \left(\frac{1}{3}\left[r^3\right]_0^{\sqrt{2}} - \frac{1}{3}\left[(4-r^2)^{3/2}\right]_{\sqrt{2}}^2 \right) \\
&= \frac{2\pi}{3}\left(2\sqrt{2} + 2\sqrt{2}\right) \\
&= \frac{8\sqrt{2}\,\pi}{3}
\end{aligned}$$