Name: ____________________

Clear your desk of everything excepts pens, pencils and erasers.
If you have a question, please raise your hand.

1. (2 points) Write the point $(x, y, z) = (\sqrt{3}, 1, 1)$ in cylindrical coordinates (r, θ, z).

Solution: The z-coordinate stays the same. So we just need to convert $(\sqrt{3}, 1)$ into polar coordinates. The length of this vector is $\sqrt{3+1} = 2$, so $r = 2$. Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, this means that $\cos(\theta) = \frac{\sqrt{3}}{2}$ and $\sin(\theta) = \frac{1}{2}$, and so $\theta = \frac{\pi}{6}$.

2. (2 points) Write the point $(x, y, z) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 1\right)$ in spherical coordinates (ρ, θ, ϕ).

Solution: The length of the vector (x, y, z) is $\rho = \sqrt{\frac{1}{4} + \frac{3}{4} + 1} = \sqrt{2}$. Forgetting the z-coordinate, the projection to the x, y-plane is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and so $\theta = \frac{\pi}{3}$. Finall,y ϕ is the angle between the vectors $\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 1 \right\rangle$ and $\langle 0, 0, 1\rangle$. The dot product of these vectors is 1, so the angle is $\phi = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.

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3. (3 points) Use a triple integral in cylindrical coordinates to compute the volume of the region between the two paraboloids $z = x^2 + y^2$ and $z = 2 - (x^2 + y^2)$.

Solution: The equations of the two surfaces in cylindrical coordinates are $z = r^2$ and $z = 2 - r^2$. Setting them equal, we see that they intersect when $r = 1$ and $z = 1$. So the volume is given by

$$\begin{aligned}
\iiint_E dx\,dy\,dz &= \int_0^1 \int_0^{2\pi} \int_{r^2}^{2-r^2} r\,dz\,d\theta\,dr \\
&= \int_0^1 \int_0^{2\pi} r(2 - r^2 - r^2)\,d\theta\,dr \\
&= 2\int_0^1 \int_0^{2\pi} r(1 - r^2)\,d\theta\,dr \\
&= 4\pi \int_0^1 (r - r^3)\,dr \\
&= \pi
\end{aligned}$$