Name: ______________________________

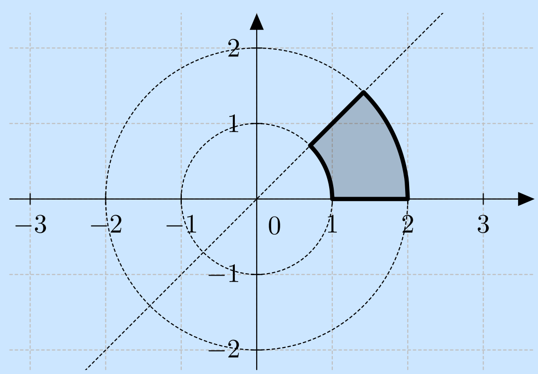
Clear your desk of everything excepts pens, pencils and erasers.
If you have a question, please raise your hand.

1. (2 points) Let D be the region of the plane described by the inequalities:

   - $1 \leq x^2 + y^2 \leq 4$
   - $x \geq 0$
   - $0 \leq y \leq x$

   Which is the proper way to convert the integral $\iint_D \left(1 + xy + \frac{1}{x^2+y^2}\right) dx\, dy$ to polar coordinates?

Solution: For the integrand function, just substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The differential $dx\, dy$ becomes $r\, dr\, d\theta$. For the limits on the integrals, we must examine the region D. It looks like:



We can see that D is described by the polar inequalities $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{4}$. So the transformed integral is

$$\int_0^{\pi/4} \int_1^2 \left(1 + r^2\cos(\theta)\sin(\theta) + \frac{1}{r^2}\right) r\, dr\, d\theta$$

Continue on to back side

2. (2 points) What is the volume of the region above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 1$?

Solution: The two surfaces intersect in the circle of radius 1 (in the plane $z = 1$). So the volume is given by integrating the top minus the bottom over the region inside the unit circle D:

$$\text{volume} = \iint_D \left(1 - \sqrt{x^2 + y^2}\right) dx\, dy$$

This is simpler in polar coordinates, where it becomes

$$\begin{aligned}\int_0^1 \int_0^{2\pi} (1 - r) r \, d\theta \, dr &= 2\pi \int_0^1 (r - r^2)\, dr \\ &= 2\pi \left(\frac{1}{2} - \frac{1}{3}\right) \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3}\end{aligned}$$

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3. (2 points) Consider the 3-dimensional region E which is above the paraboloid $z = 4(x^2 + y^2)$ and below the plane $z = 4$. Suppose the density in this region is given by $\rho(x, y, z) = z$. Compute the mass of the region E.

Solution: We know that mass is computed as the integral of density:

$$\text{mass of } E = \iiint_E \rho(x,y,z)\,dV$$

The intersection of the top and bottom surfaces $z = 4(x^2 + y^2)$ and $z = 4$ is the circle of radius 1 in the plane $z = 4$. So the "shadow"/projection of E onto the x, y-plane is the inside of the unit circle. So we can write our integral as

$$\begin{aligned}\iiint_E \rho(x,y,z)\,dV &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{4(x^2+y^2)}^{4} z\,dz\,dy\,dx \\ &= \frac{1}{2}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(16 - 16(x^2+y^2)^2\right)\,dy\,dx \\ &= 8\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - (x^2+y^2)^2)\,dy\,dx\end{aligned}$$

At this point, it would be easiest to convert to polar coordinates:

$$\begin{aligned}8\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - (x^2+y^2)^2)\,dy\,dx &= 8\int_0^1\int_0^{2\pi} (1 - r^4)r\,d\theta\,dr \\ &= 16\pi\int_0^1 (r - r^5)dr \\ &= 16\pi\left(\frac{1}{2} - \frac{1}{6}\right) \\ &= \frac{16\pi}{3}\end{aligned}$$