Name: ____________________

Clear your desk of everything excepts pens, pencils and erasers.
If you have a question, please raise your hand.

1. (2 points) Consider the function $f(x,y) = \frac{y}{1+x^2+y^2}$. Its partial derivatives are:

$$f_x = \frac{-2xy}{(1+x^2+y^2)^2} \qquad f_y = \frac{1+x^2-y^2}{(1+x^2+y^2)^2}$$

$$f_{xx} = \frac{2y(3x^2-y^2-1)}{(1+x^2+y^2)^3} \quad f_{yy} = \frac{-2y(3x^2-y^2+3)}{(1+x^2+y^2)^3}$$

$$f_{xy} = \frac{-2x(x^2-3y^2+1)}{(1+x^2+y^2)^3}$$

Where does f have a local maximum?

Solution: Notice that f_x can only be zero if either x or y is zero (since the denominator can never be zero). But looking at f_y, we see that if $y = 0$, then f_y can not be zero, since $1 + x^2$ is never zero. But, if $x = 0$, then $1 - y^2$ can be zero if $y = \pm 1$. So we see there are two critical points where $\nabla f = \langle 0, 0 \rangle$, at $(0, 1)$ and $(0, -1)$.

Now use the second derivative test at the two points. First do $(x, y) = (0, 1)$. Then the second derivatives are $f_{xx} = -\frac{1}{2}$, $f_{yy} = -\frac{1}{2}$, and $f_{xy} = 0$. So the Hessian determinant is

$$D(0,1) = f_{xx}(0,1) \cdot f_{yy}(0,1) - f_{xy}(0,1)^2 = \frac{1}{4}$$

Since $D > 0$, $(0, 1)$ is not a saddle point. Since $f_{xx} < 0$, it is a local maximum. So the answer is $(0, 1)$.

Continue on to back side

2. (2 points) What is the tangent plane to the surface $xyz + \sin(x+y+z) = 1 + \dfrac{\pi^3}{288}$ at the point $\left(\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}\right)$?

Solution: Define the function $f(x, y, z) = xyz + \sin(x + y + z)$. Then the surface above is given by $f(x, y, z) = 1 + \frac{\pi^3}{288}$. The normal vector to the tangent plane is then given by the gradient ∇f. Compute the partial derivatives to see that

$$\begin{aligned}\nabla f(x, y, z) &= \langle yz + \cos(x+y+z),\ xz + \cos(x+y+z),\ xy + \cos(x+y+z)\rangle \\ &= \langle yz, xz, xy\rangle + \cos(x+y+z)\,\langle 1, 1, 1\rangle\end{aligned}$$

We are interested in the point $(x, y, z) = \left(\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}\right)$:

$$\begin{aligned}\nabla f\left(\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}\right) &= \left\langle \frac{\pi^2}{24}, \frac{\pi^2}{48}, \frac{\pi^2}{72}\right\rangle + \cos\left(\frac{\pi}{2}\right)\langle 1, 1, 1\rangle \\ &= \frac{\pi^2}{24}\left\langle 1, \frac{1}{2}, \frac{1}{3}\right\rangle\end{aligned}$$

We can rescale and just use $\langle 1, 2, 3\rangle$ as the normal vector for our plane. So the equation is

$$\left(x - \frac{\pi}{12}\right) + \frac{1}{2}\left(y - \frac{\pi}{6}\right) + \frac{1}{3}\left(z - \frac{\pi}{4}\right) = 0$$

3. (2 points) Compute the double integral

$$\iint_R \sin(6x - y)\, dA$$

where R is the rectangle with $0 \le x \le \frac{\pi}{2}$ and $0 \le y \le \frac{\pi}{2}$.

Solution: We can integrate in whichever order we like. Let's do x first:

$$\begin{aligned}
\int_0^{\pi/2} \int_0^{\pi/2} \sin(6x - y) dx\, dy &= -\frac{1}{6} \int_0^{\pi/2} \left[\cos(6x - y)\right]_0^{\pi/2} dy \\
&= -\frac{1}{6} \int_0^{\pi/2} \left(\cos(3\pi - y) - \cos(-y)\right) dy \\
&= -\frac{1}{6} \int_0^{\pi/2} \left(-\cos(y) - \cos(-y)\right) dy \qquad (1) \\
&= -\frac{1}{6} \int_0^{\pi/2} \left(-\cos(y) - \cos(y)\right) dy \qquad (2) \\
&= \frac{1}{3} \int_0^{\pi/2} \cos(y) dy \\
&= \frac{1}{3} \left[\sin(y)\right]_0^{\pi/2} \\
&= \frac{1}{3}
\end{aligned}$$

The above labelled steps have used the facts:

1. $\cos(\pi + t) = -\cos(t)$
   Visually, this is the fact that if you go halfway around the circle, the x-value is negated.
2. $\cos(-t) = \cos(t)$
   This is because $\cos(t)$ is an even function. Visually, if you reflect the circle over the x-axis, the x-value doesn't change.

An alternate solution (integrating in the other order):

Solution:

$$\begin{aligned}
\int_0^{\pi/2}\int_0^{\pi/2} \sin(6x-y)dy\,dx &= \int_0^{\pi/2} [\cos(6x-y)]_0^{\pi/2}\,dx \\
&= \int_0^{\pi/2} \left(\cos\left(6x-\frac{\pi}{2}\right)-\cos(6x)\right)\,dx \\
&= \frac{1}{6}\left[\sin\left(6x-\frac{\pi}{2}\right)-\sin(6x)\right]_0^{\pi/2} \\
&= \frac{1}{6}\left(\sin\left(\frac{5\pi}{2}\right)-\sin\left(-\frac{\pi}{2}\right)-\sin(3\pi)+\sin(0)\right) \\
&= \frac{1}{6}\left(\sin\left(\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)-\sin(\pi)+\sin(0)\right) \\
&= \frac{1}{6}(1+1-0+0) \\
&= \frac{1}{3}
\end{aligned}$$