Name: ____

1. Consider the vector field

$$\vec{F}(x, y, z) = \left\langle \sin(y),\ x\cos(y) + \cos(z),\ \frac{1}{1+z^2} - y\sin(z) \right\rangle$$

(a) Find a potential function for $\vec{F}$.
(i.e. find a function $f(x, y, z)$ so that $\nabla f = \vec{F}$)

Solution: Integrate the first component with respect to x:

$$f = \int f_x\, dx = \int \sin(y)\, dx = x\sin(y) + g(y, z)$$

for some function g depending on y and z. Now differentiate this with respect to y:

$$f_y = x\cos(y) + g_y$$

On the other hand, it should be equal to $x\cos(y) + \cos(z)$, so $g_y = \cos(z)$. Integrate with respect to y to get g:

$$g = \int g_y\, dy = \int \cos(z)\, dy = y\cos(z) + h(z)$$

for some function h depending only on z. Combining with the above results, we get

$$f = x\sin(y) + y\cos(z) + h(z)$$

Differentiate with respect to z:

$$f_z = -y\sin(z) + h'(z)$$

But it should also be equal to $\frac{1}{1+z^2} - y\sin(z)$, so we see that $h'(z) = \frac{1}{1+z^2}$. Integrate to see that $h(z) = \tan^{-1}(z) + c$ for any choice of constant c. So we finally get

$$f = x\cos(y) + y\cos(z) + \tan^{-1}(z) + c$$

(b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where the curve C is given parametrically by $\vec{r}(t) = \left\langle \cos(t), \sin(t), \frac{t}{\pi} \right\rangle$ for $0 \leq t \leq \pi$.

Solution: By part (a), the vector field $\vec{F}$ is conservative ($\vec{F} = \nabla f$), so we can use the **Fundamental Theorem for Line Integrals**, which says that

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(\pi)) - f(\vec{r}(0))$$

At $t = 0$, we have $\vec{r}(0) = \langle 1, 0, 0 \rangle$, and so $f(\vec{r}(0)) = \tan^{-1}(0) = 0$.
At $t = \pi$, we have $\vec{r}(\pi) = \langle -1, 0, 1 \rangle$, and so $f(\vec{r}(\pi)) = \tan^{-1}(1) = \frac{\pi}{4}$.
The value of the line integral is then $\frac{\pi}{4} - 0 = \frac{\pi}{4}$.

2. Consider the triangle with vertices $(1, 2, 3)$, $(1, 0, 1)$, and $(-1, 1, 0)$.

(a) Compute the area of the triangle.

Solution: Let's choose the vertex $(1, 0, 1)$, and write the vectors $\vec{v}$ and $\vec{w}$ which point from this vertex to the other two:

$$\vec{v} = \langle 1, 2, 3 \rangle - \langle 1, 0, 1 \rangle = \langle 0, 2, 2 \rangle$$

$$\vec{w} = \langle -1, 1, 0 \rangle - \langle 1, 0, 1 \rangle = \langle -2, 1, -1 \rangle$$

Then their cross product is $\vec{v} \times \vec{w} = \langle -4, -4, 4 \rangle = 4 \langle -1, -1, 1 \rangle$. The area of the triangle is half of the length of this vector:

$$\text{area} = \frac{1}{2} |\vec{v} \times \vec{w}| = 2\sqrt{3}$$

(b) Find an equation of the plane which contains the triangle.

Solution: From part (a), we know that $\langle -1, -1, 1 \rangle$ is a normal vector to the plane. Let's choose a point we know is on the plane, for example $(1, 2, 3)$. Then the equation of the plane is

$$-(x - 1) - (y - 2) + z - 3 = 0$$

This can be re-written as $z = x + y$.

(c) Compute the line integral $\int_C \langle 2z, 5x, y\rangle \cdot d\vec{R}$ where C is the boundary of the triangle, traversed counter-clockwise when viewed from above.

Solution: Since the curve is closed, let's use **Stokes' Theorem**:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

So we need to compute the curl of $\vec{F}$ and choose a surface S whose boundary is C. First, let's compute the curl

$$\nabla \times \vec{F} = \langle 1, 2, 5\rangle$$

For the surface S, let's use the flat triangle considered above. In other words, it is the part of the plane $z = x + y$ which lies above the triangle D in the x, y-plane with vertices $(1, 2)$, $(1, 0)$, and $(-1, 1)$. Before we do the computation, we need to compute

$$d\vec{S} = \langle -z_x, -z_y, 1\rangle = \langle -1, -1, 1\rangle \, dA$$

So using **Stokes' Theorem**, we get

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \iint_S \nabla \times \vec{F} \cdot d\vec{S} \\ &= \iint_D \langle 1, 2, 5\rangle \cdot \langle -1, -1, 1\rangle \, dA \\ &= 2 \iint_D dA\end{aligned}$$

This is just twice the area of D. Since D is a triangle with base 2 and height 2, its area is 2. So the value of the line integral is 4.

3. Find and classify all the critical points of $f(x, y) = x^2 + y^2 + x^2y$.

Solution: First compute the gradient:

$$\nabla f = \left\langle 2x(1+y),\, 2y + x^2 \right\rangle$$

We see there are two possible ways for f_x to be zero: either $x = 0$ or $y = -1$. On the other hand, in order for f_y to be zero, we must have $y = -\frac{1}{2}x^2$. In the case that $x = 0$, we get $y = -\frac{1}{2}(0)^2 = 0$. In the case that $y = -1$, we get $2 = x^2$, and so $x = \pm\sqrt{2}$. So there are three critical points: $(0, 0)$, $(-\sqrt{2}, -1)$, and $(\sqrt{2}, -1)$.

Now compute the Hessian matrix of second partial derivatives

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2(1+y) & 2x \\ 2x & 2 \end{pmatrix}$$

To use the **Second Derivative Test**, we need to compute the determinant of this matrix:

$$D = f_{xx}f_{yy} - f_{xy}^2 = 4(1+y) - 4x^2 = 4(1 + y - x^2)$$

At $(0, 0)$, $D = 4 > 0$, and $f_{xx} = 2 > 0$, so $(0, 0)$ is a local minimum.
At $(-\sqrt{2}, -1)$, $D = -8 < 0$, so $(-\sqrt{2}, -1)$ is a saddle point.
At $(\sqrt{2}, -1)$, $D = -8 < 0$, so $(\sqrt{2}, -1)$ is a saddle point.

4. Evaluate the integral $\iiint\limits_E x^2\,dV$, where E is the region inside the sphere $x^2+y^2+z^2=1$, and between the cones $z^2=\frac{1}{3}(x^2+y^2)$ and $z^2=3(x^2+y^2)$.

Solution: Notice that after substituting the spherical change-of-variables, the equation for a cone of the form $z^2=c(x^2+y^2)$ becomes $\tan(\phi)=\frac{1}{\sqrt{c}}$. In this example, we have $c=3$ and $c=\frac{1}{3}$. Then $c=3$ corresponds to $\tan(\phi)=\frac{1}{\sqrt{3}}$, and so $\phi=\pi/6$. Also, $c=\frac{1}{3}$ corresponds to $\tan(\phi)=\sqrt{3}$, and so $\phi=\pi/3$. So in spherical coordinates, the region E is decribed by the inequalities:

$$0\le\rho\le 1$$

$$0\le\theta\le 2\pi$$
$$\frac{\pi}{6}\le\phi\le\frac{\pi}{3}$$

So the integral becomes

$$\begin{aligned}
\iiint\limits_E x^2\,dV &= \int_{\pi/6}^{\pi/3}\int_0^{2\pi}\int_0^1 (\rho\sin(\phi)\cos(\theta))^2\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi \\
&= \int_{\pi/6}^{\pi/3}\int_0^{2\pi}\int_0^1 \rho^4\cos^2(\theta)\sin^3(\phi)\,d\rho\,d\theta\,d\phi \\
&= \frac{1}{5}\int_{\pi/6}^{\pi/3}\int_0^{2\pi}\cos^2(\theta)\sin^3(\phi)\,d\theta\,d\phi \\
&= \frac{\pi}{5}\int_{\pi/6}^{\pi/3}\sin^3(\phi)\,d\phi \\
&= \frac{\pi}{5}\left[\cos(\phi)\left(\frac{1}{3}\cos^2(\phi)-1\right)\right]_{\pi/6}^{\pi/3} \\
&= \frac{\pi}{5}\cdot\frac{1}{8}\left(3\sqrt{3}-\frac{11}{3}\right) \\
&= \frac{\pi}{40}\left(3\sqrt{3}-\frac{11}{3}\right)
\end{aligned}$$

5. Suppose that $f = x^2y + e^{x-y}$, and $x = t + 3s$ and $y = s^2 - t^2$. If $t = 1$ and $s = 2$, then what is $\dfrac{\partial f}{\partial t}$?

Solution: Plug in $t = 1$ and $s = 2$ to get $x = 7$ and $y = 3$. Next Differentiate x and y, and substitute:

$$x_t = 1, \quad y_t = -2t = -2$$

Now differentiate f and substitute:

$$f_x = 2xy + e^{x-y} = 42 + e^4$$

$$f_y = x^2 - e^{x-y} = 49 - e^4$$

Finally, put it all together with the chain rule:

$$f_t = f_x \cdot x_t + f_y \cdot y_t = 42 + e^4 - 2(49 - e^4)$$

6. Compute the angle between the vectors $\vec{v} = \langle 1, 1, 1 \rangle$ and $\vec{w} = \frac{1}{2} \left\langle 0, \sqrt{3} - \sqrt{5}, \sqrt{3} + \sqrt{5} \right\rangle$.

Solution: We need to use the formula $\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}$. The dot product is

$$\vec{v} \cdot \vec{w} = \frac{1}{2} \left(\sqrt{3} - \sqrt{5} + \sqrt{3} + \sqrt{5} \right) = \sqrt{3}$$

The length of $\vec{v}$ is $\sqrt{3}$, and the length of $\vec{w}$ is

$$|\vec{w}| = \frac{1}{2} \sqrt{3 + 5 + 2\sqrt{15} + 3 + 5 - 2\sqrt{15}} = \frac{1}{2} \sqrt{16} = 2$$

So we get that $\cos(\theta) = \frac{1}{2}$. Then the angle is $\theta = \frac{\pi}{3}$.

7. Find a unit vector which is orthogonal to both $\langle 1, 2, -3 \rangle$ and $\langle 0, 0, 1, \rangle$.

Solution: Take the cross product:

$$\langle 1, 2, -3 \rangle \times \langle 0, 0, 1 \rangle = \langle 2, -1, 0 \rangle$$

This is orthogonal to the original two vectors. Now rescale to make it unit length, to get $\frac{1}{\sqrt{5}} \langle 2, -1, 0 \rangle$.

8. Suppose you are moving along the path $\vec{r}(t) = \left\langle \frac{1}{t^2+1}, \frac{t}{t^2+1} \right\rangle$. If you start at the point $(1, 0)$ and move along this path a total distance of $\frac{\pi}{4}$, what position do you end up at?

Solution: Take the derivative of $\vec{r}$:

$$\vec{r}\,'(t) = \left\langle \frac{-2t}{(t^2+1)^2}, \frac{1-t^2}{(t^2+1)^2} \right\rangle = \frac{1}{(t^2+1)^2}\left\langle -2t, 1-t^2 \right\rangle$$

Now take the length of this:

$$\begin{aligned} |\vec{r}\,'(t)| &= \frac{\sqrt{(2t)^2+(1-t)^2}}{(t^2+1)^2} \\ &= \frac{\sqrt{t^2+2t+1}}{(t^2+1)^2} \\ &= \frac{t^2+1}{(t^2+1)^2} \\ &= \frac{1}{t^2+1} \end{aligned}$$

The arclength function is then the integral of this:

$$s(T) = \int_0^T \frac{1}{t^2+1}\,dt = \tan^{-1}(T)$$

We want to know where we end up when $s = \frac{\pi}{4}$. We can see that this corresponds to $t = 1$. So we end up at the point

$$\vec{r}(1) = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$$