Name: ____________________

1. Sketch the domain of the function $f(x, y) = \sqrt{y} + \ln(25 - x^2 - y^2)$.

   (Indicate with solid or dashed lines whether the boundary points are included)

2. (5 points) Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3y^2\cos^2(x)}{x^2 + 2y^2}$ if it exists, or show it does not exist.

Solution: Let's try approaching along a generic line $y = mx$:

$$\begin{aligned}\lim_{x\to 0} \frac{3(mx)^2\cos^2(x)}{x^2 + 2(mx)^2} &= \lim_{x\to 0} \frac{3m^2x^2\cos^2(x)}{x^2 + 2m^2x^2} \\ &= \lim_{x\to 0} \frac{3m^2\cos^2(x)}{1 + 2m^2} \\ &= \frac{3m^2}{1 + 2m^2} \cdot \lim_{x\to 0} \cos^2(x) \\ &= \frac{3m^2}{1 + 2m^2}\end{aligned}$$

Since this expression depends on m, the limit does not exist.

3. Evaluate the double integrals

(a) (6 points) $\int_0^1 \int_y^1 e^{-x^2} \; dx \; dy$

Solution: Switch the order of integration:

$$\begin{aligned} \int_0^1 \int_y^1 e^{-x^2} \, dx \, dy &= \int_0^1 \int_0^x e^{-x^2} \, dy \, dx \\ &= \int_0^1 x e^{-x^2} \, dx \\ &= -\frac{1}{2} \int_0^{-1} e^u \, du && (u = -x^2) \\ &= \frac{1}{2} \int_{-1}^0 e^u \, du \\ &= \frac{1}{2} \left(1 - \frac{1}{e}\right) \end{aligned}$$

(b) (6 points) $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{-(x^2+y^2)} \; dy \; dx$

Solution: Change to polar coordinates:

$$\begin{aligned} \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{-(x^2+y^2)} \, dy \, dx &= \int_0^1 \int_0^{2\pi} e^{-r^2} r \, d\theta \, dr \\ &= 2\pi \int_0^1 e^{-r^2} r \, dr \\ &= -\pi \int_0^{-1} e^u \, du && (u = -r^2) \\ &= \pi \int_{-1}^0 e^u \, du \\ &= \pi \left(1 - \frac{1}{e}\right) \end{aligned}$$

4. (5 points) Find the length of the curve given by $\vec{r}(t) = \left\langle \frac{2}{3}\, t^{3/2},\, \cos(2t),\, \sin(2t) \right\rangle$ between $t = 0$ and $t = 5$.

Solution:

$$\vec{r}\,'(t) = \left\langle \sqrt{t},\, -2\sin(2t),\, 2\cos(2t) \right\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{t+4}$$

So the length of the curve is

$$\begin{aligned} \int_0^5 \sqrt{t+4}\, dt &= \int_4^9 \sqrt{u}\, du && (u = t+4) \\ &= \frac{2}{3}\left[u^{3/2}\right]_4^9 \\ &= \frac{2}{3}(27-8) \\ &= \frac{38}{3} \end{aligned}$$

5. (16 points) Find the area of the part of the surface $z - xy = 5$ that lies within the cylinder $x^2 + y^2 = 3$.

Solution: Call D the circle of radius $\sqrt{3}$ in the x, y-plane. Then the surface area is

$$\begin{aligned} \iint_D \sqrt{1 + x^2 + y^2}\, dA &= \int_0^{\sqrt{3}} \int_0^{2\pi} \sqrt{1+r^2}\, r\, d\theta\, dr \\ &= 2\pi \int_0^{\sqrt{3}} \sqrt{1+r^2}\, r\, dr \\ &= \pi \int_1^4 \sqrt{u}\, du && (u = 1 + r^2) \\ &= \frac{2\pi}{3}\left[u^{3/2}\right]_1^4 \\ &= \frac{2\pi}{3}(8-1) \\ &= \frac{14\pi}{3} \end{aligned}$$

6. (12 points) Evaluate $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = \left\langle x^2 y,\, xy^2 + \frac{2}{3}x^3 \right\rangle$ and C is the positively oriented circle $x^2 + y^2 = 1$.

Solution: Use **Green's Theorem**:

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= \iint_D \left(y^2 + 2x^2 - x^2 \right) dA \\
&= \iint_D \left(y^2 + x^2 \right) dA \\
&= \int_0^1 \int_0^{2\pi} r^3 \, d\theta \, dr \\
&= 2\pi \int_0^1 r^3 \, dr \\
&= \frac{\pi}{2}
\end{aligned}$$

7. (16 points) Evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = \langle x^3, y^3, z^3 \rangle$ and S is the sphere $x^2 + y^2 + z^2 = 1$, oriented outward.

Solution: Use the **Divergence Theorem**:

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \iiint_E \nabla \cdot \vec{F} \, dV \\
&= 3 \iiint_E \left(x^2 + y^2 + z^2 \right) dV \\
&= 3 \int_0^1 \int_0^{\pi} \int_0^{2\pi} \rho^4 \sin(\phi) \, d\theta \, d\phi \, d\rho \\
&= 6\pi \int_0^1 \int_0^{\pi} \rho^4 \sin(\phi) \, d\phi \, d\rho \\
&= 12\pi \int_0^1 \rho^4 \, d\rho \\
&= \frac{12\pi}{5}
\end{aligned}$$

8. (20 points) Find the volume of the solid enclosed by $z = x^2 + y^2$ and $z = 6 - x^2 - y^2$.

Solution: The two surfaces meet in a circle of radius $\sqrt{3}$, so the projection D onto the x, y-plane is the disc of radius $\sqrt{3}$. Then the volume is

$$\begin{aligned} \iint_D \left(6 - 2(x^2+y^2)\right)\, dA &= 2 \iint_D \left(3 - (x^2+y^2)\right)\, dA \\ &= 2 \int_0^{\sqrt{3}} \int_0^{2\pi} \left(3 - r^2\right) r \, d\theta \, dr \\ &= 4\pi \int_0^{\sqrt{3}} \left(3r - r^3\right)\, dr \\ &= 4\pi \left(\frac{3}{2} \left[r^2\right]_0^{\sqrt{3}} - \frac{1}{4} \left[r^4\right]_0^{\sqrt{3}} \right) \\ &= 4\pi \left(\frac{9}{2} - \frac{9}{4} \right) \\ &= 9\pi \end{aligned}$$

9. You want to make a cardboard box with no top, and volume 32 cubic inches. What dimensions will give the smallest possible surface area?

Solution: Call the dimensions x, y, and z. Since the volume is 32, we have $z = \frac{32}{xy}$. The surface area is given by

$$A = xy + 2(xz + yz) = xy + 64\left(\frac{1}{x} + \frac{1}{y}\right)$$

The gradient of $A(x, y)$ is

$$\nabla A = \left\langle y - \frac{64}{x^2},\, x - \frac{64}{y^2} \right\rangle$$

The critical points of A occur when both $y = \frac{64}{x^2}$ and $x = \frac{64}{y^2}$. Combining the equations gives $x = 4$, which then implies that $y = 4$ and $z = 2$.