Name: ______________________________

1. (2 points) Compute the surface integral $\iint_S f(x,y,z)\,dS$ where $f(x,y,z) = z + x^2y$, and S is the part of the cylinder $y^2 + z^2 = 1$ for which $y \geq 0,\ z \geq 0$, and $0 \leq x \leq 3$.

Solution: Let's parameterize S with a type of cylindrical coordinates where we leave x alone and change y, z to polar:

$$\vec{r}(x,\theta) = \langle x, \cos(\theta), \sin(\theta)\rangle$$

Here we'd have $0 \leq x \leq 3$ and $0 \leq \theta \leq \frac{\pi}{2}$. Take derivatives to get

$$\vec{r}_x = \langle 1, 0, 0\rangle$$

$$\vec{r}_\theta = \langle 0, -\sin(\theta), \cos(\theta)\rangle$$

Take the cross product:

$$\vec{r}_x \times \vec{r}_\theta = \langle 0, -\cos(\theta), -\sin(\theta)\rangle$$
$$|\vec{r}_x \times \vec{r}_\theta| = 1$$

So the original integral becomes

$$\begin{aligned}\iint_S (z + x^2y)\,dS &= \int_0^{\pi/2}\int_0^3 (\sin(\theta) + x^2\cos(\theta))\,dx\,d\theta \\ &= \int_0^{\pi/2} (3\sin(\theta) + 9\cos(\theta))\ d\theta \\ &= -3\left[\cos(\theta)\right]_0^{\pi/2} + 9\left[\sin(\theta)\right]_0^{\pi/2} \\ &= 12\end{aligned}$$

Continue on to back side

2. (2 points) Compute the surface integral $\iint\limits_S \vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle ze^{xy},\ -3ze^{xy}, xy\rangle$, and S is the parallelogram with vertices $(0,0,1)$, $(1,-1,2)$, $(2,2,5)$, and $(3,1,6)$, with upward orientation.

Choose either part (a) or part (b). You can get a bonus point if you do both.

(a) Compute by thinking of the parallelogram as part of a graph $z = f(x,y)$.

Solution: To use this method, we need to find the equation of the plane which contains the parallelogram. Let's take two edges with a common vertex and write them as vectors. For example, let's take the edges which meet at the vertex $(0,0,1)$. Then the vectors are $\langle 1,-1,1\rangle$ and $\langle 2,2,4\rangle$. The normal vector to the plane is their cross-product:

$$\langle 1,-1,1\rangle \times \langle 2,2,4\rangle = \langle -6,-2,4\rangle$$

So the plane is given by the equation $-6x - 2y + 4z = 4$, or we can solve for z to get

$$z = 1 + \frac{3}{2}x + \frac{1}{2}y$$

Our $d\vec{S}$ is given by

$$d\vec{S} = \langle -z_x, -z_y, 1\rangle\, dx\, dy = \left\langle -\frac{3}{2}, -\frac{1}{2}, 1\right\rangle dx\, dy$$

Now plug everything in and the integral becomes

$$\begin{aligned}\iint\limits_S \vec{F}\cdot d\vec{S} &= \iint\limits_D \left\langle \left(1+\frac{3}{2}x+\frac{1}{2}y\right)e^{xy}, -3\left(1+\frac{3}{2}x+\frac{1}{2}y\right)e^{xy}, xy\right\rangle \cdot \left\langle -\frac{3}{2}, -\frac{1}{2}, 1\right\rangle dx\, dy \\ &= \iint\limits_D xy\, dx\, dy\end{aligned}$$

Here, D is the region in the x,y-plane under the parallelogram. So it is the parallelogram with vertices $(0,0)$, $(1,-1)$, $(2,2)$, and $(3,1)$. Unfortunately, we have to break this into 3 parts: where $0 \le x \le 1$, where $1 \le x \le 2$, and where $2 \le x \le 3$:

$$\begin{aligned}\iint\limits_D xy\, dx\, dy &= \int_0^1 \int_{-x}^{x} xy\, dy\, dx + \int_1^2 \int_{x-2}^{x} xy\, dy\, dx + \int_2^3 \int_{x-2}^{4-x} xy\, dy\, dx \\ &= 0 + \frac{5}{3} + \frac{7}{3} \\ &= 4\end{aligned}$$

(b) Compute using the parameterization $\vec{r}(u,v) = \langle u+v, u-v, 1+2u+v\rangle$ for $0 \le u \le 2$, and $0 \le v \le 1$.

Solution: Compute the normal vector:

$$\vec{r}_u = \langle 1, 1, 2\rangle$$
$$\vec{r}_v = \langle 1, -1, 1\rangle$$
$$\vec{r}_u \times \vec{r}_v = \langle 3, 1, -2\rangle$$

We want *upward* orientation, so we need to use $\vec{r}_v \times \vec{r}_u = \langle -3, -1, 2\rangle$. The surface integral becomes

$$\begin{aligned}\iint_S \vec{F}\cdot d\vec{S} &= \int_0^1 \int_0^2 \left\langle (1+2u+v)e^{u^2-v^2}, -3(1+2u+v)e^{u^2-v^2}, u^2-v^2 \right\rangle \cdot \langle -3, -1, 2\rangle \, du\, dv \\ &= 2\int_0^1\int_0^2 (u^2-v^2)\, du\, dv \\ &= 2\int_0^1 \left(\frac{8}{3} - 2v^2\right) dv \\ &= 4\end{aligned}$$

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