Name: ______________________________

Clear your desk of everything excepts pens, pencils and erasers.
If you have a question, please raise your hand.

1. (2 points) If C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$, traversed counter-clockwise, then what is the value of the line interal $\int\limits_C (y\cos(x)+2y)\,dx+\sin(x)\,dy$?

Solution: By **Green's Theorem**, this line integral is equal to

$$\begin{aligned}\iint\limits_D \left(\frac{\partial}{\partial x}(\sin(x)) - \frac{\partial}{\partial y}(y\cos(x)+2y)\right) dA &= \iint\limits_D (\cos(x) - (\cos(x)+2))\,dA \\ &= -2\iint\limits_D dA \\ &= -2\cdot \text{area}(D)\end{aligned}$$

The area of the triangle D is $\frac{1}{2}$, so the value of the integral is -1.

Continue on to back side

2. (2 points) Evaluate the integral $\int_C 2xe^{-2y}\,dx + \left(10y - 2x^2e^{-2y}\right)\,dy$, where C is a path from $(2,0)$ to $(3,2)$.

Solution: Let's check if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ to see if the field is conservative.

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(10y - 2x^2e^{-2y}) = -4xe^{-2y}$$
$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xe^{-2y}) = -4xe^{-2y}$$

They are the same, so the vector field is conservative. We will try to find a potential function.

$$f = \int f_x\,dx = \int 2xe^{-2y}\,dx = x^2e^{-2y} + g(y)$$

Now differentiate with respect to y, and compare with the original vector field:

$$f_y = -2xe^{-2y} + g'(y) = 10y - 2x^2e^{-2y}$$

From this, we get that $g'(y) = 10y$, and so $g(y) = 5y^2 + c$ for a constant c. So finally,

$$f = 5y^2 + x^2e^{-2y} + c$$

Now use the fundamental theorem for line integrals:

$$\begin{aligned}\int_C 2xe^{-2y}\,dx + \left(10y - 2xe^{-2y}\right)\,dy &= \int_C \nabla f\,d\vec{r}\\ &= f(3,2) - f(2,0)\\ &= \left(20 + 9e^{-4}\right) - (4)\\ &= 16 + 9e^{-4}\end{aligned}$$

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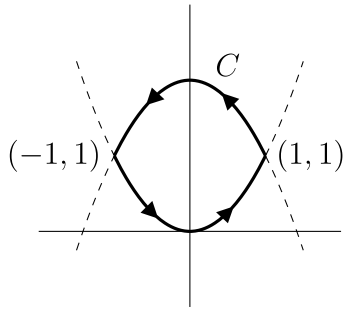
3. (2 points) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F}(x,y) = \left\langle 2y + e^{\sqrt{x}},\ 3x + \cos(y^4) \right\rangle$$

and C follows $y = x^2$ from $(-1,1)$ to $(1,1)$, and then follows $y = 2 - x^2$ from $(1,1)$ back to $(-1,1)$:



(Hint: try using Green's Theorem)

Solution: Since this is a closed curve, we can use **Green's Theorem**. The line integral is equal to

$$\begin{aligned}\iint_D \left(\frac{\partial}{\partial x}(3x + \cos(y^4)) - \frac{\partial}{\partial y}(2y + e^{\sqrt{x}}) \right) dA &= \iint_D (3-2)\, dA \\ &= \iint_D dA \\ &= \int_{-1}^{1} ((2-x^2) - x^2)\, dx \\ &= 2\int_{-1}^{1} (1-x^2)\, dx \\ &= 2\left(2 - \frac{2}{3}\right) \\ &= 4 - \frac{4}{3} \\ &= \frac{8}{3}\end{aligned}$$