Name: ______________________________

1. (2 points) Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = \langle xy,\ 3y^2 \rangle$, and the curve C is parameterized by $\vec{r}(t) = \langle 11t^4,\ t^3 \rangle$ for $0 \leq t \leq 1$.

Solution: First take the derivative:

$$\vec{r}\,'(t) = \left\langle 44t^3, 3t^2 \right\rangle$$

Then substitute : $\vec{F} = \langle xy, 3y^2 \rangle = \langle 11t^7, 3t^6 \rangle$. Finally,

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \left\langle 11t^7, 3t^6 \right\rangle \cdot \left\langle 44t^3, 3t^2 \right\rangle dt \\ &= \int_0^1 \left(484t^{10} + 9t^8\right) dt \\ &= \left(44\left[t^{11}\right]_0^1 + \left[t^9\right]_0^1\right) \\ &= 45 \end{aligned}$$

Continue on to back side

2. (2 points) Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = \langle 2xe^{-y},\, 2y - x^2e^{-y} \rangle$, and C is the upper-left quarter of a circle of radius 1, centered at $(2, 0)$. The curve C can be described by $\vec{r}(t) = \langle 2 - \cos(t),\, \sin(t) \rangle$ for $0 \leq t \leq \frac{\pi}{2}$.

Solution: Let's check whether $\vec{F}$ is conservative.

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(2y - x^2e^{-y}\right) = -2xe^{-y}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(2xe^{-y}\right) = -2xe^{-y}$$

Se we can conclude that yes, $\vec{F}$ is conservative. That means there is some "potential function" f so that $\nabla f = \vec{F}$, and then our integral is just equal to $f(2, 1) - f(1, 0)$. Now let's find the potential function.

$$f(x, y) = \int f_x(x, y)\, dx = \int 2xe^{-y}\, dx = x^2e^{-y} + g(y)$$

for some function $g(y)$ which depends only on y. Now let's differentiate with respect to y. From what we just got, this should be

$$f_y = -x^2e^{-y} + g'(y)$$

On the other hand, since ∇f should be equal to $\vec{F}$, this should also be

$$f_y = 2y - x^2e^{-y}$$

So we conclude that $g'(h) = 2y$, meaning $g(y) = y^2$. So we finally get that the potential function is

$$f(x, y) = y^2 + x^2e^{-y}$$

Now, by the fundamental theorem for line integrals, we get

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(2, 1) - f(1, 0) = 1 + \frac{4}{e} - 1 = \frac{4}{e}$$