Example 5.8(d) Evaluate the indefinite integral $\dfrac{\sin(t)}{\cos^2(\cos(t))}dt$

Solution: We can re-write $\frac{\sin(t)}{\cos^2(\cos(t))}$ as $\sec^2(\cos(t)) \cdot \sin(t)$. Let's make the substitution:

$$u = \cos(t)$$
$$du = -\sin(t)dt$$

Now we can write the integral as

$$\begin{aligned}\int \frac{\sin(t)}{\cos^2(\cos(t))}dt &= -\int \sec^2(\cos(t))(-\sin(t))dt \\ &= -\int \sec^2(u)du \\ &= \tan(u) + C \\ &= \tan(\cos(t)) + C\end{aligned}$$

Example 5.11 Evaluate the following indefinite integrals:

(WW2) $\displaystyle\int \frac{x^3}{\sqrt{5+9x^4}}dx$

Solution: Let's make the substitution

$$u = 9x^4 + 5$$
$$du = 36x^3 dx$$

If we both multiply and divide the integral by 36, we get

$$\begin{aligned}\int \frac{x^3}{\sqrt{9x^4+5}}dx &= \frac{1}{36}\int \frac{36x^3}{\sqrt{9x^4+5}}dx \\ &= \frac{1}{36}\int \frac{du}{\sqrt{u}} \\ &= \frac{1}{36} \cdot 2\sqrt{u} + C \\ &= \frac{1}{18}\sqrt{9x^4+5} + C\end{aligned}$$

1. $\displaystyle\int \frac{9\sin(\sqrt{x})}{\sqrt{x}}dx$

Solution: Let's make the substitution:

$$u = \sqrt{x}$$
$$du = \frac{1}{2\sqrt{x}}dx$$

If we both multiply and divide the integral by 2, we get

$$\begin{aligned}\int \frac{9\sin(\sqrt{x})}{\sqrt{x}}dx &= 18\int \frac{\sin(\sqrt{x})}{2\sqrt{x}}dx \\ &= 18\int \sin(u)du \\ &= -18\cos(u) + C \\ &= -18\cos(\sqrt{x}) + C\end{aligned}$$

2. $\displaystyle\int \frac{1}{x^2}\sin\left(\frac{3}{x}\right)\cos\left(\frac{3}{x}\right)dx$

Solution: Let's make the substitution

$$\begin{aligned}u &= \sin\left(\frac{3}{x}\right) \\ du &= -\frac{3}{x^2}\cos\left(\frac{3}{x}\right)dx\end{aligned}$$

Multiplying and dividing by -3, we get

$$\begin{aligned}\int \frac{1}{x^2}\sin\left(\frac{3}{x}\right)\cos\left(\frac{3}{x}\right)dx &= -\frac{1}{3}\int \sin\left(\frac{3}{x}\right)\cos\left(\frac{3}{x}\right)\cdot\left(-\frac{3}{x^2}\right)dx \\ &= -\frac{1}{3}\int u du \\ &= -\frac{1}{6}u^2 + C \\ &= -\frac{1}{6}\sin^2\left(\frac{3}{x}\right) + C\end{aligned}$$

3. $\displaystyle\int 5x\sqrt{x-4}dx$

Solution: Let's make the substitution

$$\begin{aligned}u &= x-4 \\ du &= dx\end{aligned}$$

Noticing from the above equation that $x = (x-4)+4 = u+4$, we get

$$\begin{aligned}\int 5x\sqrt{x-4}dx &= 5\int (x-4+4)\sqrt{x-4}\,dx \\ &= 5\int (u+4)\sqrt{u}\,du \\ &= 5\int u\sqrt{u}\,du + 20\int \sqrt{u}\,du \\ &= 2u^{5/2} + \frac{40}{3}u^{3/2} + C \\ &= 2(x-4)^{5/2} + \frac{40}{3}(x-4)^{3/2} + C\end{aligned}$$

Example 5.12 Evaluate the following definite integrals by changing the limits of integration appropriately:

(WW12) $\int_{-4}^{2} \frac{dx}{\sqrt{9-2x}} dx$

Solution: Let's make the substitution

$$u = 9 - 2x$$
$$du = -2dx$$

We additionally need to change the limits of integration:

$$x = -4 \longrightarrow u = 17$$
$$x = 2 \longrightarrow u = 5$$

Multiplying and dividing by -2, we get

$$\begin{aligned}\int_{-4}^{2} \frac{dx}{\sqrt{9-2x}} &= -\frac{1}{2}\int_{-4}^{2} \frac{-2dx}{\sqrt{9-2x}} \\ &= -\frac{1}{2}\int_{17}^{5} \frac{du}{\sqrt{u}} \\ &= \frac{1}{2}\int_{5}^{17} \frac{du}{\sqrt{u}} \\ &= \frac{1}{2}\left[2\sqrt{u}\right]_5^{17} \\ &= \sqrt{17} - \sqrt{5}\end{aligned}$$

(WW13) $\int_{\pi/6}^{\pi/2} \frac{\cos(z)}{\sin^{5/2}(z)} dz$

Solution: Let's make the substitution (we'll use x instead of u, since we aren't using x for anything right now)

$$x = \sin(z)$$
$$dx = \cos(z)dz$$

We'll also change the limits of integration:

$$z = \frac{\pi}{6} \longrightarrow x = \frac{1}{2}$$
$$z = \frac{\pi}{2} \longrightarrow x = 1$$

Now we make the substitutions in the integral:

$$\begin{aligned}\int_{\pi/6}^{\pi/2} \frac{\cos(z)dz}{(\sin(z))^{5/2}} &= \int_{1/2}^{1} \frac{dx}{x^{5/2}} \\ &= -\frac{2}{3}\left[x^{-3/2}\right]_{1/2}^{1} \\ &= \frac{2}{3}\left[x^{-3/2}\right]_{1}^{1/2} \\ &= \frac{2}{3}\left(2^{3/2} - 1\right)\end{aligned}$$

Example 5.13 Evaluate the following definite integrals:

(WW15) $\int_{-\pi/16}^{\pi/16} (4x + \sin(8x))dx$

Solution: This is an odd function, and we are integrating over an interval centered at zero, so the integral is zero.

(WW16) $\int_{-4}^{4} \frac{t^3}{13 + 2t^2} dt$

Solution: This is an odd function, and we are integrating over an interval centered at zero, so the integral is zero.