Example 4.10 Find the most general function $F(x)$ with the property that:

(a) $F'(x) = 3x^{2/3} - \dfrac{1}{x^2}$

Solution: $F(x) = \displaystyle\int F'(x)dx = \int \left(3x^{2/3} - \frac{1}{x^2}\right) dx = \frac{9}{5}x^{5/3} + \frac{1}{x} + C$

(b) $F'(x) = \sec(x)\,(\tan(x) + \sec(x))$

Solution: $F(x) = \displaystyle\int F'(x)dx = \int \left(\sec(x)\tan(x) + \sec^2(x)\right) dx = \sec(x) + \tan(x) + C$

(c) $F'(x) = 3x(x^2 - 5x + 2)$

Solution: $F(x) = \displaystyle\int F'(x)dx = \int \left(3x^3 - 15x^2 + 6x\right) dx = \frac{3}{4}x^4 - 5x^3 + 3x^2 + C$

Example 4.11 Find the net change of the function $f(x)$ from $x = 0$ to $x = 1$, if $f'(x) = x^2 + x + 1$.

Solution: The net change is the integral of the derivative:

$$\begin{aligned}
\text{net change} &= f(1) - f(0) \\
&= \int_0^1 f'(x)dx \\
&= \int_0^1 \left(x^2 + x + 1\right) \\
&= \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_0^1 \\
&= \frac{1}{3} + \frac{1}{2} + 1 \\
&= \frac{11}{6}
\end{aligned}$$

Example 4.12 Find the average value of the given function on the given interval. Also, find the value of c guaranteed by the **Mean Value Theorem for Integrals**.

(a) $f(x) = \sqrt[3]{x}$ on $[0, 8]$

Solution:

$$\begin{aligned} \text{average value} &= \frac{1}{8}\int_0^8 x^{1/3}dx \\ &= \frac{1}{8} \cdot \frac{3}{4}\left[x^{4/3}\right]_0^8 \\ &= \frac{3}{32} \cdot 8^{4/3} \\ &= \frac{3}{32} \cdot 16 \\ &= \frac{3}{2} \end{aligned}$$

(b) $f(x) = \sec^2(x)$ on $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

Solution:

$$\begin{aligned} \text{average value} &= \frac{1}{\frac{\pi}{4} - \frac{-\pi}{4}}\int_{-\pi/4}^{\pi/4} \sec^2(x)dx \\ &= \frac{2}{\pi}\left[\tan(x)\right]_{-\pi/4}^{\pi/4} \\ &= \frac{2}{\pi}\left(1 - (-1)\right) \\ &= \frac{4}{\pi} \end{aligned}$$