Example 3.9 Evaluate the following integrals

(a) $\displaystyle\int_1^8 \frac{dx}{\sqrt[3]{x^2}}$

Solution:

$$\begin{aligned}
\int_1^8 \frac{dx}{\sqrt[3]{x^2}} &= \int_1^8 x^{-2/3}dx \\
&= \left[3x^{1/3}\right]_1^8 \\
&= \left[\sqrt[3]{x}\right]_1^8 \\
&= 2-1 \\
&= 1
\end{aligned}$$

(b) $\displaystyle\int_0^{\pi/3} \sin(\theta)d\theta$

Solution:

$$\begin{aligned}
\int_0^{\pi/3} \sin(\theta)d\theta &= [-\cos(\theta)]_0^{\pi/3} \\
&= -\cos\left(\frac{\pi}{3}\right) + \cos(0) \\
&= -\frac{1}{2} + 1 \\
&= \frac{1}{2}
\end{aligned}$$

(c) $\displaystyle\int_{-2}^{2} f(x)dx$, where $f(x) = \begin{cases} 2 & \text{if } \; -2 \leq x \leq 0 \\ 4 - x^2 & \text{if } \; 0 < x \leq 2 \end{cases}$

Solution:

$$\begin{aligned}
\int_{-2}^{2} f(x)dx &= \int_{-2}^{0} 2dx + \int_0^2 (4-x^2)dx \\
&= 4 + \left[4x - \frac{x^3}{3}\right]_0^2 \\
&= 4 + 8 - \frac{8}{3} \\
&= 12 - \frac{8}{3} \\
&= \frac{28}{3}
\end{aligned}$$

(d) $\displaystyle\int_{-1}^{5} |3x-6|\,dx$

Solution: Since $|3x-6|=0$ when $x=2$, we have

$$\begin{aligned}\int_{-1}^{5} |3x-6|\,dx &= \int_{-1}^{2}(6-3x)dx + \int_{2}^{5}(3x-6)dx \\ &= \left[6x-\frac{3}{2}x^2\right]_{-1}^{2} + \left[\frac{3}{2}x^2-6x\right]_{2}^{5} \\ &= \left(12-6+6+\frac{3}{2}\right) + \left(\frac{75}{2}-30-6+12\right) \\ &= \frac{78}{2}-12 \\ &= 39-12 \\ &= 27\end{aligned}$$

Example 3.10 Identify what is wrong with the evaluation:

$$\begin{aligned}\int_{-1}^{1}\frac{1}{x^2}dx &= \left[-x^{-1}\right]_{-1}^{1} \\ &= \left(-1^{-1}+(-1)^{-1}\right) \\ &= -1-1 \\ &= -2\end{aligned}$$

Solution: The **Fundamental Theorem of Calculus** only applies if the function is continuous on the interval $[a,b]$, which in this case is $[-1,1]$. But the function $\frac{1}{x^2}$ is not continuous at $x=0$, so the theorem does not apply.

Example 3.11 Find the derivatives of the following functions:

(a) $F(x) = \displaystyle\int_{2}^{1/x} \sin^4(t)dt$

Solution: If we let $f(x) = \displaystyle\int_{2}^{x}\sin^4(t)dt$, then the **Fundamental Theorem** says that

$$f'(x) = \sin^4(x)$$

Since $F(x) = f\left(\frac{1}{x}\right)$, the chain rule says that

$$\begin{aligned}F'(x) &= f'\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right) \\ &= -\frac{1}{x^2}\sin^4\left(\frac{1}{x}\right)\end{aligned}$$

(b) $G(x) = \displaystyle\int_{\sin(x)}^{1}\cos^2(\theta)d\theta$

Solution: If we let $g(x) = \int_x^1 \cos^2(\theta)d\theta = -\int_1^x \cos^2(\theta)d\theta$, then the **Fundamental Theorem** says that

$$g'(x) = -\cos^2(x)$$

Since $G(x) = g(\sin(x))$, the chain rule says that

$$\begin{aligned} G'(x) &= g'(\sin(x)) \cdot \cos(x) \\ &= -\cos^2(\sin(x)) \cdot \cos(x) \end{aligned}$$

(c) $H(x) = \int_{\tan(x)}^{x^2} \frac{dt}{\sqrt{2+t^4}}$

Solution: If we let $h(x) = \int_0^x \frac{dt}{\sqrt{2+t^4}}$, then the **Fundamental Theorem** says that

$$h'(x) = \frac{1}{\sqrt{2+x^4}}$$

Since $H(x) = h\left(x^2\right) - h(\tan(x))$, the chain rule says that

$$\begin{aligned} H'(x) &= h'\left(x^2\right) \cdot 2x - h'(\tan(x)) \cdot \sec^2(x) \\ &= \frac{2x}{\sqrt{2+x^{16}}} - \frac{\sec^2(x)}{\sqrt{2+\tan^4(x)}} \end{aligned}$$