Example 1.9 Solve the initial value problem

$$f'(x) = 1 + 3\sqrt{x}$$
$$f(4) = 25$$

Solution. To find f, we merely take the antiderivative of $f'(x)$, and we see that

$$f(x) = x + 2x^{3/2} + C$$

We can use the initial condition ($f(4) = 25$) to solve for C:

$$\begin{aligned} 25 &= f(4) \\ 25 &= 4 + 2(4)^{3/2} + C \\ 25 &= 4 + 2(\sqrt{4})^3 + C \\ 25 &= 4 + 2(2)^3 + C \\ 25 &= 20 + C \\ 5 &= C \end{aligned}$$

So the solution is the function:

$$f(x) = x + 2x^{3/2} + 5$$

Example 1.10 Solve the initial value problem

$$f''(x) = 2 + \cos(x)$$
$$f(0) = -1$$
$$f(\pi/2) = 0$$

Solution. First, we take the antiderivative to get $f'(x)$:

$$f'(x) = 2x + \sin(x) + C$$

Now we take the antiderivative again to get $f(x)$:

$$f(x) = x^2 - \cos(x) + Cx + B$$

Let's see what the first initial condition ($f(0) = -1$) tells us:

$$\begin{aligned} -1 &= f(0) \\ -1 &= 0^2 - \cos(0) + C(0) + B \\ -1 &= -1 + B \\ 0 &= B \end{aligned}$$

So the function actually looks like

$$f(x) = x^2 + Cx - \cos(x)$$

We can use the second initial condition ($f(\pi/2) = 0$) to solve for C:

$$\begin{aligned}
0 &= f(\pi/2) \\
0 &= \left(\frac{\pi}{2}\right)^2 + \frac{C\pi}{2} - \cos\left(\frac{\pi}{2}\right) \\
0 &= \frac{\pi^2}{4} + \frac{C\pi}{2} \\
-\frac{\pi^2}{4} &= \frac{C\pi}{2} \\
-\frac{\pi}{2} &= C
\end{aligned}$$

The solution is the function

$$f(x) = x^2 - \frac{\pi}{2}x - \cos(x)$$