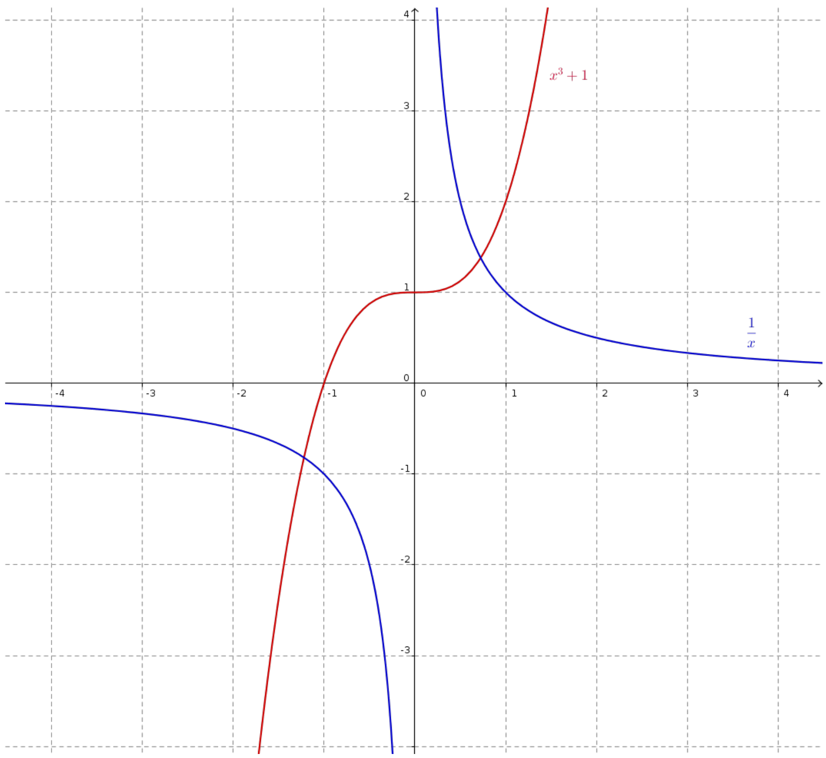
**Example 9.8** Use Newton's method to approximate all roots of the equation $\frac{1}{x} = 1 + x^3$. Sketch these graphs and use the **Intermediate Value Theorem** to select the best integer values for starting points. Stop at $x_2$.

**Solution.** If we graph the functions $x^3 + 1$ and $\frac{1}{x}$, we see:




We can see that there are two intersection points — one is between $-2$ and $-1$, and the other is between 0 and 1. Define the function

$$f(x) = x^3 + 1 - \frac{1}{x}$$

The intersection points we are looking for are the zeros of $f(x)$. The derivative is given by

$$f'(x) = 3x^2 + \frac{1}{x^2}$$

First let's approximate the left intersection point (which is a zero of $f(x)$), using $x_1 = -1$ for our initial guess:

$$x_2 = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{1}{4} = -\frac{5}{4}$$

Now let's approximate the right intersection point, using $x_1 = 1$ as our initial guess:

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{4} = \frac{3}{4}$$

**Example 9.9** Use Newton's method to approximate $\sqrt[3]{29}$. Calculate up to $x_3$. Do not simplify.

**Solution.** We will use the function $f(x) = x^3 - 29$, with initial guess $x_1 = 3$ (since $3^3 = 27$):

$$x_2 = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{-2}{27} = 3 + \frac{2}{27} = \frac{83}{27}$$

$$x_3 = \frac{83}{27} - \frac{\left(\frac{83}{27}\right)^3 - 29}{3\left(\frac{83}{27}\right)^2}$$

**Example 9.10** Find the most general antiderivative of the functions below:

$(a)$ $f(x) = (3 + 2x)^2$

**Solution.** If we multiply it out, we get

$$f(x) = (3 + 2x)^2 = 9 + 12x + 4x^2$$

The antiderivative is then

$$9x + 6x^2 + \frac{4}{3}x^3 + C$$

$(b)$ $f(x) = \dfrac{3 - x + 5x^3}{x^3}$

**Solution.** We can re-write $f$ as:

$$f(x) = \frac{3}{x^3} - \frac{1}{x^2} + 5 = 3x^{-3} - x^{-2} + 5$$

The antiderivative is then

$$-\frac{3}{2}x^{-2} + x^{-1} + 5x + C = -\frac{3}{2x^2} + \frac{1}{x} + 5x + C$$