Example 3.6 For the following functions, find the intervals on which it is increasing and decreasing, and find where the local maximum and local minimum values occur.

(*a*) $f(x) = 2x^3 + 3x^2 - 36x$ on the domain $(-\infty, \infty)$

Solution. The derivative is

$$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x + 3)(x - 2)$$

The graph of $f'(x)$ is a parabloa that opens up, so it is positive *outisde* of the x-intercepts, and negative *between* the x-intercepts. So the function $f(x)$ is increasing on

$$(-\infty, -3) \cup (2, \infty)$$

and it is decreasing on

$$(-3, 2)$$

The derivative is zero at $x = -3$ and $x = 2$, so these are local extrema. Using the first derivative test, we see that $x = -3$ is a local maximum and $x = 2$ is a local minimum.

(*b*) $f(x) = \cos^2(x) - 2\sin(x)$ on the domain $[0, 2\pi]$

Solution. The derivative is

$$f'(x) = -2\cos(x)\sin(x) - 2\cos(x) = -2\cos(x)\left(\sin(x) + 1\right)$$

We see that $f'(x) = 0$ when either $\cos(x) = 0$ or $\sin(x) + 1 = 0$. In the first case, $\cos(x) = 0$ when $x = \pi/2$ or $x = \frac{3\pi}{2}$. In the second case, we have $\sin(x) + 1 = 0$ when $x = \frac{3\pi}{2}$. Evaluate $f'(x)$ at any point in $\left(0, \frac{\pi}{2}\right)$ to see that f' is negative here, and so f is decreasing. Evaluate $f'(x)$ at any point in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ to see that f' is positive here, and so f is increasing. Finally, evaluate $f'(x)$ at any point in $\left(\frac{3\pi}{2}, 2\pi\right)$ to see that f' is negative here, and so f is decreasing. The first derivative test tells us that $x = \frac{\pi}{2}$ is a local minimum and $x = \frac{3\pi}{2}$ is a local maximum.

Example 3.7 For each of the following functions (on the given domain), tell whether the hypotheses of the **Mean Value Theorem** are satisfied. If so, try to find the value of c guaranteed by the theorem.

(*a*) $f(x) = \dfrac{1}{x}$ on the domain $[1, 3]$

Solution. The function $\frac{1}{x}$ is continuous and differentiable everywhere except at $x = 0$, so the hypotheses of the theorem are satisfied. The theorem says that there is some number c between 1 and 3 so that

$$f'(c) = \frac{\frac{1}{3} - \frac{1}{1}}{3 - 1} = \frac{-1}{3}$$

We know that the derivative is $f'(x) = \frac{-1}{x^2}$, we can solve algebraically:

$$\begin{aligned} \frac{-1}{c^2} &= \frac{-1}{3} \\ c^2 &= 3 \\ c &= \sqrt{3} \end{aligned}$$

Technically, $c = -\sqrt{3}$ is also a solution to this equation, but it is not in the interval $[1, 3]$.

(*b*) $f(x) = x^3 - 3x + 2$ on the domain $[-2, 2]$

Solution. This is a polynomial, so it is continuous and differentiable *everywhere*, and so the hypotheses of the theorem are satisfied. The theorem tells us that there is a number c in between -2 and 2 so that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{4 - 0}{4} = 1$$

We know that $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$, and so we can solve for c algebraically:

$$\begin{aligned} 3(c^2 - 1) &= 1 \\ c^2 - 1 &= \frac{1}{3} \\ c^2 &= \frac{4}{3} \\ c &= \pm\frac{2}{\sqrt{3}} \end{aligned}$$

Example 3.8 Suppose that $f(x)$ is continuous and differentiable everywhere, with $f(8) = 30$, and $f'(x) \leq 1$ for all x. What is the largest that $f(10)$ could be?

Solution. Since f is continuous and differentiable everywhere, it is in particular continuous and differentiable on the interval $[8, 10]$, and so we can apply the **Mean Value Theorem** on that interval. The theorem says that there is a number c with $8 < c < 10$ so that the derivative at c is equal to the average rate of change over the interval $[8, 10]$:

$$f'(c) = \frac{f(10) - f(8)}{10 - 8} = \frac{f(10) - 30}{2}$$

We can re-arrange this equation and solve for $f(10)$ to get

$$f(10) = 2f'(c) + 30$$

We know that $f'(x) \leq 1$ for all x, so $f'(c) \leq 1$ as well. This gives us that

$$f(10) \leq 2(1) + 30 = 32$$

Example 3.9 Prove that the equation $2x + \cos(x) = 0$ has *exactly one* solution.

Solution. Let f be the function $f(x) = 2x + \cos(x)$. We want to show that $f(x) = 0$ has one (and only one) solution. First we'll show that it has *at least* one solution. Notice that if we plug in $x = 0$ and $x = -\pi$, we get

$$f(0) = 2(0) + \cos(0) = 1 > 0$$

$$f(-\pi) = 2(-\pi) + \cos(-pi) = -\pi - 1 < 0$$

Since $f(-\pi) < 0 < f(0)$, the **Intermediate Value Theorem** says that there is some c with $-\pi \leq c \leq 0$ so that $f(c) = 0$. So $x = c$ is one solution to our equation. Now let's see that this is the *only* solution.

What if there was another solution? Suppose that $x = a$ is another number (different from c) such that $f(a) = 0$. Then by **Rolle's Theorem** (or the **Mean Value Theorem**) there is a number b in between a and c with $f'(b) = \frac{f(c)-f(a)}{c-a} = \frac{0-0}{c-a} = 0$. But if we differentiate $f(x)$, we see

$$f'(x) = 2 - \sin(x)$$

Since $\sin(x)$ is always in between -1 and 1, this means that $f'(x) \geq 1$ for all values of x. In particular, $f'(x)$ is *never* equal to zero. But we just said that $f'(b) = 0$. So there cannot exist such a number b. This contradiction came from our assumption that there was a second solution $x = a$, so that must not have been valid.

Example 3.10 For the following functions, find the intervals on which it is increasing and decreasing, and find where the local maximum and local minimum values occur.

(a) $f(x) = \dfrac{x}{x^2+1}$ on the domain $(-\infty, \infty)$

Solution. The derivative is given by

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \frac{1 - x^2}{(1 + x^2)^2}$$

The critical points are at $x = -1$ and $x = 1$. Since the denominator is always positive, we just need to look at the numerator to determine if $f'(x)$ is positive or negative. We see that $f'(x)$ is positive if $x^2 < 1$, and it is negative if $x^2 > 1$. In other words, $f'(x)$ is positive on the interval $(-1, 1)$ and negative on $(-\infty, -1) \cup (1, \infty)$. So f is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1) \cup (1, \infty)$. By the first derivative test, $x = -1$ is a local minimum and $x = 1$ is a local maximum.

(b) $f(x) = \sin(x) + \cos(x)$ on the domain $[0, 2\pi]$

Solution. The derivative is

$$f'(x) = \cos(x) - \sin(x)$$

The critical points are where $f'(x) = 0$, which is where $\cos(x) = \sin(x)$. Since cosine and sine are the coordinates of points on the unit circle, the points where they are equal are the points where the unit circle intersects the line $y = x$. This happens at the two angles $\pi/4$ and $5\pi/4$. So these are the critical points of f in the interval $[0, 2\pi]$.

From 0 to $\pi/4$, we have $\cos(x) \geq \sin(x)$, and so $f'(x) \geq 0$ on $\left[0, \frac{\pi}{4}\right]$. So f is decreasing on this interval.

From $\pi/4$ to $5\pi/4$, we have $\sin(x) \geq \cos(x)$, and so $f'(x) \leq 0$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$. So f is increasing on this interval.

From $5\pi/4$ to 2π, we have $\cos(x) \geq \sin(x)$, and so $f'(x) \geq 0$ on $\left[\frac{5\pi}{4}, 2\pi\right]$. So f is decreasing on this interval.

By the first derivative test, $x = \pi/4$ is a local minimum, and $x = 5\pi/4$ is a local maximum.