Example 1.12 Find the critical numbers for the function $f(x) = \sqrt{1-x^2}$.

Solution. First differentiate:

$$f'(x) = \frac{x}{\sqrt{1-x^2}}$$

We see that this is zero at $x = 0$, so zero is one critical number. We also have critical numbers at $x = 1$ and $x = -1$, since the function is defined here, but it is not differentiable ($f'(x)$ is undefined at $x = \pm 1$).

Example 1.14 Find the absolute maximum and minimum values of $f(x) = 12 + 4x - x^2$ on the interval $[0, 5]$.

Solution. The derivative is

$$f'(x) = 4 - 2x$$

We see that $f'(x) = 0$ when $x = 2$, so this is a critical number. Now we just evaluate f at $x = 2$ as well as the endpoints:

$$f(0) = 12$$

$$f(2) = 16$$

$$f(5) = 7$$

The absolute maximum value is 16, which occurs at $x = 2$. The absolute minimum value is 7, which occurs at $x = 5$.

Example 1.15 Find the absolute maximum and minimum values of $f(x) = \dfrac{x}{x^2 - x + 1}$ on the interval $[0, 3]$.

Solution. The derivative is

$$f'(x) = \frac{(x^2 - x + 1) - x(2x - 1)}{(x^2 - x + 1)^2} = \frac{1 - x^2}{(x^2 - x + 1)^2}$$

We see that $f'(x) = 0$ when $x = -1$ or $x = 1$. So these are critical numbers (although -1 is not in $[0, 3]$). So we just have to evaluate f at $0, 1,$ and 3.

$$f(0) = 0$$

$$f(1) = 1$$

$$f(3) = \frac{3}{7}$$

The absolute maximum value is 1, which occurs at $x = 1$. The absolute minimum value is 0, which occurs at $x = 0$.

Example 1.16 Find the absolute maximum and minimum values of $f(x) = 2\cos(t) + \sin(2t)$ on the interval $[0, \pi/2]$.

Solution. The derivative is

$$f'(x) = 2\cos(2t) - 2\sin(t) = 2(\cos(2t) - \sin(t))$$

To find the critical number(s), we solve $f'(x) = 0$:

$$\begin{aligned} 2(\cos(2t) - \sin(t)) &= 0 \\ \cos(2t) - \sin(t) &= 0 \\ \cos(2t) &= \sin(t) \end{aligned}$$

In the interval $[0, \pi/2]$, we see that $f'(x) = 0$ only when $x = \pi/6$, since $\sin(\pi/6) = \cos(\pi/3) = 1/2$. So we evaluate f at $x = \pi/6$ and at the endpoints:

$$f(0) = 2$$

$$f(\pi/6) = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

$$f(\pi/2) = 0$$

The absolute maximum value is $\frac{3\sqrt{3}}{2}$, which occurs at $x = \pi/6$. The absolute minimum value is 0, which occurs at $x = \pi/2$.