**Example 8.11** Let  $f(x) = x - x^2 = x(1 - x)$ , and let  $\theta$  be the angle between the positive x-axis and the line joining the point (x, f(x)) with the origin. At what rate is  $\theta$  changing, with respect to x, when  $x = 1 - \frac{1}{\sqrt{3}}$ ?



**Solution.** We see from the picture that there is a right triangle where the vertical side has length f(x) and the horizontal side has length x. These sides are opposite and adjacent to the angle  $\theta$ , so we can write

$$\tan(\theta) = \frac{f(x)}{x} = 1 - x$$

Differentiating this equation (with respect to x), we get

$$\sec^2(\theta) \cdot \frac{d\theta}{dx} = -1$$

Since  $\sec(\theta) = \frac{1}{\cos(\theta)}$ , we can multiply both sides by  $\cos^2(\theta)$  to get

$$\frac{d\theta}{dx} = -\cos^2(\theta)$$

Now, to evaluate this, we just need to figure out what  $\theta$  is when  $x = 1 - \frac{1}{\sqrt{3}}$ . Going back to our original equation, we have

$$\tan(\theta) = 1 - x = 1 - \left(1 - \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$$

From this we can see that  $\theta = \frac{\pi}{6}$ , and so

$$\frac{d\theta}{dx} = -\cos^2\left(\frac{\pi}{6}\right) = -\left(\frac{\sqrt{3}}{2}\right)^2 = -\frac{3}{4}$$

**Example 8.12** A person is lifting a weight with a pulley. The pulley is 25 feet off the ground, the rope is 45 feet long, and the person is holding the end of the rope 5 feet off the ground. If the person is walking backwards (away from the pulley) at a constant rate of 5 ft/sec, how fast is the weight rising when the person is 10 feet from the spot on the ground directly under the pulley?



**Solution.** Let x be the distance from the person to the spot under the pulley. If you draw a horizontal line going through the person's hands, and a vertical line coming down from the pulley, you will get a right triangle, where the horizontal side has length x, and the vertical side has length 20 (since the pulley is 25 feet up, and the person's hands are 5 feet up).

Let's figure out what the length of the hypotenuse of this triangle is. We know the rope is 45 feet long. The entire rope is composed of the hypotenuse of our triangle, plus the little vertical bit going from the pulley to the weight. Let's call the vertical distance between the height of the weight and the height of the person's hands h. Then the vertical part of the rope has length 20 - h. Therefore the length of the hypotenuse is 45 - (20 - h) = h + 25. Now, using the Pythagorean Theorem, we get

$$x^2 + 20^2 = (h + 25)^2$$

Differentiating (with respect to t), we get

$$2x \cdot x' = 2(h+25) \cdot h'$$

We are asked to find how fast the weight is rising, so we need to solve for h':

$$h' = \frac{x \cdot x'}{h + 25}$$

Now we plug in everything we know. The problem says that x' = 5, and we want to find x' when x = 10. We just need to find the value of h to plug in. But we can use our triangle and the Pythagorean theorem to solve for h:

$$10^2 + 20^2 = (h + 25)^2$$

We get that  $h = \sqrt{500} - 25$ . Plugging everything in, we get

$$h' = \frac{(10)(5)}{\sqrt{500}} = \sqrt{5}$$

**Example 8.13** A spotlight on the ground shines on a wall 12 meters away. If a man 2 meters tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

**Solution.** Let x be the distance from the man to the light. Then the distance from the man to the wall is 12 - x. We have similar triangles, so  $\frac{2}{x} = \frac{h}{12}$ , if h is the height of the shadow. Cross-multiplying gives xh = 24. Differentiating gives:

$$\begin{aligned} x'\cdot h + x\cdot h' &= 0\\ h' &= -\frac{x'\cdot h}{x} \end{aligned}$$

We know x' = 1.6, and when the man is 4 feet from the building, we have x = 8, and so

$$h' = -\frac{1.6h}{8} = -0.2h$$

When x = 8, we have h = 3, and so h' = -0.6.