Example 6.8 The height (in meters) of a projective shot vertically upward from a point 2 m above the ground level with an initial velocity of 25 m/s is

$$h(t) = 2 + 25t - 5t^2$$

(a) Find the velocity after 2 seconds.

(b) When does the projective reach its maximum height?

(c) What is the maximum height?

(d) When does it hit the ground?

(e) With what velocity does it hit the ground?

Solution.

(a) The velocity is just the *derivative* of the position function, so the velocity is

$$h'(t) = 25 - 10t$$

To get the velocity after 2 seconds, we plug in $t = 2$:

$$h'(2) = 25 - 10(2) = 5$$

So the velocity is 5 meters per second after 2 seconds.

(b) **The maximum height occurs when the velocity is zero.**

$$\begin{aligned} h'(t) &= 0 \\ 25 - 10t &= 0 \\ 25 &= 10t \\ 2.5 &= t \end{aligned}$$

The maximum height occurs after 2.5 seconds.

(c) The function $h(t)$ describes the height, so plug the answer from part (b) into h:

$$h(2.5) = 2 + 25(2.5) - 5(2.5)^2 = 33.25$$

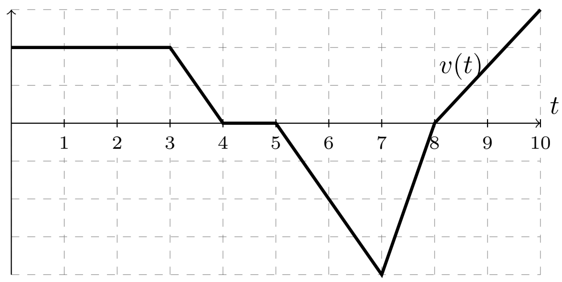
(d) It hits the ground when the height is zero. So solve the equation $h(t) = 0$. There is no calculus involved here. This is just algebra.

$$\begin{aligned} h(t) &= 0 \\ 2 + 25t - 5t^2 &= 0 \\ t &= \frac{-25 \pm \sqrt{25^2 - 4(-5)(2)}}{-10} \\ t &= 2.5 \pm \frac{\sqrt{665}}{10} \end{aligned}$$

We take the positive answer (since negative time values don't make sense).

(e) You need to plug the answer for part (d) into $h'(t)$.

Example 6.9 The figure below shows the velocity $v(t)$ of a particle moving on a horizontal coordinate line, for t in a closed interval $[0, 10]$.



(a) The particle is moving forward during:

Solution. The particle moves forward when the velocity is positive. This is happening on the time intervals $(0, 4)$ and $(8, 10)$.

(b) The particle's speed is increasing during:

Solution. The particles speed is increasing when *the absolute value* of the velocity is increasing. This means either that the velocity is either positive and increasing, or negative and decreasing (in either case, getting farther from zero). This happens on the intervals $(5, 7)$ and $(8, 10)$.

(c) The particle has positive acceleration during:

Solution. Acceleration is the derivative of velocity, so the acceleration is positive when the velocity is increasing. This happens on the interval $(7, 10)$.

(d) The particle has zero acceleration during:

Solution. The acceleration is zero on the intervals $(0, 3)$ and $(4, 5)$, since the velocity is constant on those intervals.

(e) The particle achieves its greatest speed at:

Solution. The greatest speed happens when the absolute value of the velocity is greatest. This is at $t = 7$.

(f) The particle stands still for more than an instant during:

Solution. It stands still when the velocity is zero. This happens for more than an instant on the interval $(4, 5)$.

Example 6.10 Find $\frac{dy}{dx}$ by implicit differentiation of $\tan(x-y) = \frac{y}{1+x^2}$.

Solution. This problem is much easier if you first write $\tan(x)$ as $\frac{\sin(x)}{\cos(x)}$ cross multiply before differentiating. So before taking any derivatives, re-write the equation as:

$$\begin{aligned} \tan(x-y) &= \frac{y}{1+x^2} \\ \frac{\sin(x-y)}{\cos(x-y)} &= \frac{y}{1+x^2} \\ (1+x^2)\cdot\sin(x-y) &= y\cdot\cos(x-y) \end{aligned}$$

Now, take the derivatives of both sides. Let's use the notation y' instead of $\frac{dy}{dx}$. You'll need the product rule and the chain rule:

$$2x\cdot\sin(x-y) + (1+x^2)\cdot\cos(x-y)\cdot(1-y') = y'\cdot\cos(x-y) - y\cdot\sin(x-y)\cdot(1-y')$$

This step was the only calculus in the problem. We will now solve for y', and the rest is just algebra

$$\begin{aligned} 2x\cdot\sin(x-y) + (1+x^2)\cdot\cos(x-y) + y\cdot\sin(x-y) &= \big(\cos(x-y) + y\cdot\sin(x-y) + (1+x^2)\cdot\cos(x-y)\big)\cdot y' \\ (2x+y)\cdot\sin(x-y) + (1+x^2)\cdot\cos(x-y) &= \big(y\cdot\sin(x-y) + (2+x^2)\cdot\cos(x-y)\big)\cdot y' \\ \frac{(2x+y)\cdot\sin(x-y) + (1+x^2)\cdot\cos(x-y)}{y\cdot\sin(x-y) + (2+x^2)\cdot\cos(x-y)} &= y' \end{aligned}$$

Example 6.11

(a) Find y' if $x^3 + y^3 = 6xy$.

Solution. Differentiate both sides of the equation to get

$$3x^2 + 3y^2\cdot y' = 6(y + x\cdot y')$$

Now just solve algebraically for y':

$$\begin{aligned} (3y^2 - 6x)\cdot y' &= 6y - 3x^2 \\ y' &= \frac{6y-3x^2}{3y^2-6x} \end{aligned}$$

(b) Find an equation of the tangent line at the point $(3,3)$.

Solution. To find the slope, we just plug $x = 3$ and $y = 3$ into our formula for y' that we derived in part (a).

$$y' = \frac{6(3) - 3(3)^2}{3(3)^2 - 6(3)} = -1$$

Now use the formula for the equation of the tangent line:

$$y = 3 - (x-3) = 6 - x$$

(c) At what point in the first quadrant is the tangent line horizontal?

Solution. The tangent line is horizontal when $y' = 0$. So we wolve algebraically:

$$\begin{aligned} \frac{6y - 3x^2}{3y^2 - 6x} &= 0 \\ 6y - 3x^2 &= 0 \\ 6y &= 3x^2 \\ y &= \frac{x^2}{2} \end{aligned}$$

So we must find where the curve $y = \frac{x^2}{2}$ intersects the original curve $x^3 + y^3 = 6xy$. Just plug in $y = \frac{x^2}{2}$ everywhere and solve for x:

$$\begin{aligned} x^3 + \frac{x^6}{8} &= 3x^3 \\ \frac{x^6}{8} - 2x^3 &= 0 \\ x^6 - 16x^3 &= 0 \\ x^3(x^3 - 16) &= 0 \end{aligned}$$

This would suggest that the horizontal tangents occur when $x = 0$ and $x = \sqrt[3]{16}$.