Example 8.7 A sphere is growing, its volume increasing at a constant rate of 10 in^3 per second. Let $r(t)$, $V(t)$, and $S(t)$ be the radius, volume, and surface area of the sphere at time t. If $r(1) = 2$, then compute:

(a) $r'(1)$

Solution. Relate the volume and the radius by the equation

$$V = \frac{4\pi}{3} r^3$$

Remember that everything is a function of t, so we really have

$$V(t) = \frac{4\pi}{3} r(t)^3$$

We can differentiate both sides (with respect to t):

$$V'(t) = \frac{4\pi}{3} \cdot 3r(t)^2 \cdot r'(t)$$

The right-hand side follows from the chain rule. Now solve for $r'(t)$ to get:

$$r'(t) = \frac{10}{4\pi r(t)^2}$$

Now plug in $t = 1$, and use the fact that we know $r(1) = 2$:

$$r'(1) = \frac{10}{4\pi(2)^2} = \frac{10}{16\pi} = \frac{5}{8\pi}$$

So the radius is increasing at a rate $\frac{5}{8\pi}$ inches per second at $t = 1$.

(b) $S'(1)$

Solution. First write down the equation for the surface area of a sphere

$$S = 4\pi r^2$$

Remember that everything is a function of t, so this is

$$S(t) = 4\pi r(t)^2$$

Differentiate both sides:

$$S'(t) = 4\pi \cdot 2r(t) \cdot r'(t) = 8\pi r(t) \cdot r'(t)$$

Plugging in $t = 1$, and using part (a), we get

$$S'(1) = 8\pi r(1) \cdot r'(1) = 8\pi \cdot 2 \cdot \frac{5}{8\pi} = 10$$

So the surface area is increasing at a rate of 10 square inches per second at $t = 1$.

Example 8.8 A circle is growing, its radius (in inches) give by $r(t) = \sqrt{t}$ for t in seconds. How fast is the area growing at time $t = 4$?

Solution. Write down the equation for the area of a circle:

$$A = \pi r^2$$

Remember everything is a function of t, so we have

$$A(t) = \pi r(t)^2$$

Differentiate, to get

$$A'(t) = 2\pi r(t) \cdot r'(t)$$

We want to find $A'(4)$. To do so, we need to know $r'(t)$, so differentiate:

$$r'(t) = \frac{1}{2\sqrt{t}}$$

So we get

$$A'(4) = 2\pi \cdot \sqrt{4} \cdot \frac{1}{2\sqrt{4}} = \pi$$

Example 8.9 A right triangle is growing, with its vertical side growing at a constant rate of 1 unit per second, and its horizontal side growing at a constant rate of 2 units per second. At time $t = 2$ seconds, how fast is the hypotenuse growing?

Solution. Let's call the horizontal side x and the vertical y, and the hypotenuse h. We need to find $h'(2)$. First write down the Pythagorean Theorem:

$$x^2 + y^2 = h^2$$

Now differentiate (with respect to t), to get

$$2x \cdot x' + 2y \cdot y' = 2h \cdot h'$$

Solve for h' to get

$$h' = \frac{x \cdot x' + y \cdot y'}{h}$$

We know that $x' = 1$ and $y' = 2$. Assuming that $x(t) = 2t$ and $y(t) = t$, we get that at $t = 2$,

$$h' = \frac{4 \cdot 2 + 2 \cdot 1}{\sqrt{4^2 + 2^2}} = \frac{10}{\sqrt{20}} = \sqrt{5}$$