Example 5.9 Calculate the derivatives of the following functions:

(a) $y = \sin(x\cos(x))$

(b) $y = \sqrt{x + \sqrt{x}}$

(c) $F(x) = (4x - x^2)^{100}$

Solution. (a) If we write $f(x) = \sin(x)$ and $g(x) = x\cos(x)$, then we can write y as

$$y = f \circ g(x) = f(g(x))$$

Then the chain rule says the derivative is

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

Let's calculate the derivatives of f and g. For f, it's simple. We know that

$$f'(x) = \frac{d}{dx}(\sin(x)) = \cos(x)$$

For g', we will need to use the product rule, so

$$g'(x) = 1 \cdot (\cos(x)) + x \cdot (-\sin(x)) = \cos(x) - x\sin(x)$$

Putting this all together, we get that

$$y' = \frac{dy}{dx} = \cos(x\cos(x)) \cdot (\cos(x) - x\sin(x))$$

(b) If we write $f(x) = \sqrt{x}$ and $g(x) = x + \sqrt{x}$, then, as before, we can write y as

$$y = f \circ g(x) = f(g(x))$$

Computing derivatives, we get

$$f'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$g'(x) = 1 + \frac{1}{2\sqrt{x}}$$

So by the chain rule, the derivative of y is

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= f'(g(x)) \cdot g'(x) \\ &= \frac{1}{2\sqrt{g(x)}} \cdot \left(1 + \frac{1}{2\sqrt{x}}\right) \\ &= \frac{1}{2\sqrt{x + \sqrt{x}}} \cdot \left(1 + \frac{1}{2\sqrt{x}}\right) \end{aligned}$$

(c) Let $f(x) = x^{100}$ and $g(x) = 4x - x^2$. Then we can write $F(x)$ as $f \circ g(x) = f(g(x))$. The derivatives of f and g are:

$$f'(x) = 100x^{99}$$

$$g'(x) = 4 - 2x$$

So by the chain rule,

$$\begin{aligned} F'(x) &= f'(g(x)) \cdot g'(x) \\ &= 100(g(x))^{99} \cdot (4 - 2x) \\ &= 100(4x - x^2)^{99}(4 - 2x) \end{aligned}$$

Example 5.10 Prove part **(d)** of **Theorem 4.3** (that the derivative of $\sec(x)$ is given by $\sec(x)\tan(x)$).

Solution. Recognize that $\sec(x) = \dfrac{1}{\cos(x)}$. We can then use the quotient rule to compute the derivative:

$$\begin{aligned} \frac{d}{dx}(\sec(x)) &= \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) \\ &= \frac{0 \cdot \cos(x) - 1 \cdot (-\sin(x))}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} \\ &= \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)} \\ &= \sec(x) \cdot \tan(x) \end{aligned}$$

There is another way to easily calculate the derivative of $\sec(x)$. Again, recognizing that $\sec(x) = \dfrac{1}{\cos(x)}$, we can use the chain rule to compute the derivative. If we call $f(x) = \frac{1}{x}$, then $\sec(x) = f(\cos(x))$. Then the chain rule says:

$$\begin{aligned} \frac{d}{dx}(\sec(x)) &= f'(\cos(x)) \cdot \frac{d}{dx}(\cos(x)) \\ &= \frac{-1}{\cos^2(x)} \cdot (-\sin(x)) \\ &= \frac{\sin(x)}{\cos^2(x)} \\ &= \sec(x) \cdot \tan(x) \end{aligned}$$

Examle 5.11 Prove that the derivative of an odd function is an even function.

Solution. Let $f(x)$ be an odd function. This means that

$$f(-x) = -f(x)$$

Let's simply take the derivative of both sides of this equation (we need the chain rule for the left side):

$$-f'(-x) = -f'(x)$$

Multiplying both sides by -1 gives:

$$f'(-x) = f'(x)$$

This exactly means that $f'(x)$ is an even function.

Example 5.12 A ball is expanding over time, and the radius (as a function of time in minutes) is given by

$$r(t) = \sqrt{t} + 1$$

How fast is the volume of the ball increasing at $t = 4$ minutes?

Solution. The volume of a ball is given by

$$V = \frac{4}{3}\pi r^3$$

Since r is a function of t, the chain rule says that

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

We can get $\frac{dr}{dt}$ by differentiating the formula for $r(t)$:

$$\frac{dr}{dt} = r'(t) = \frac{1}{2\sqrt{t}}$$

We can get $\frac{dV}{dr}$ by differentiating the volume formula (with respect to r):

$$\frac{dV}{dr} = V'(r) = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

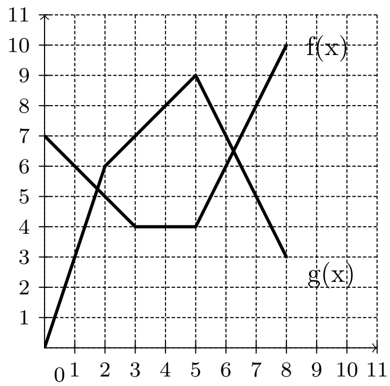
Putting it all together, we get that

$$\frac{dV}{dt} = \frac{4\pi r^2}{2\sqrt{t}} = \frac{2\pi(\sqrt{t}+1)^2}{\sqrt{t}}$$

If we evaluate at $t = 4$, we get

$$\left.\frac{dV}{dt}\right|_{t=4} = \frac{2\pi(\sqrt{4}+1)^2}{\sqrt{4}} = 9\pi$$

Example 5.13 The graphs of two functions, f and g, are pictured below.



Let $h(x) = f(g(x))$, and $p(x) = g(f(x))$. Compute the following derivatives, or explain why they do not exist:

(a) $h'(2)$

(b) $h'(7)$

(c) $p'(1)$

Solution. First, let's notice that (by the chain rule):

$$h'(x) = f'(g(x)) \cdot g'(x)$$

$$p'(x) = g'(f(x)) \cdot f'(x)$$

Even though we do not know the formulas for $f(x)$ and $g(x)$, we can see that they are piece-wise linear. Since each "piece" of the graphs is just a line, the derivative is the slope of the corresponding line.

(a) From the equations above, we have that $h'(2) = f'(g(2)) \cdot g'(2)$. But since the slopes from the left and right are different at $x = 2$, this means that $g(x)$ is not differentiable there. So $h(x)$ is also not differentiable at $x = 2$.

(b) We have that $h'(7) = f'(g(7)) \cdot g'(7)$. Looking at the graph of $g(x)$, we see that $g(7) = 5$. At $x = 7$, the graph of $g(x)$ is a line with slope -2, so $g'(7) = -2$. So we get

$$h'(7) = f'(5) \cdot (-2) = -2f'(5)$$

But the slopes of $f(x)$ from the left and right at $x =$ are different, so f is not differentiable at $x = 5$. Therefore h is also not differentiable at $x = 7$.

(c) We have that $p'(1) = g'(f(1)) \cdot f'(1)$. We see from the graph that $f(1) = 6$. At $x = 1$, the graph of $f(x)$ is a line with slope -1, so $f'(1) = -1$. So we have

$$p'(1) = g'(6) \cdot (-1) = -g'(6)$$

At $x = 6$, the graph of $g(x)$ is a line with slope -2, so $g'(6) = -2$. Then we get

$$p'(1) = -(-2) = 2$$