Example 2.8 Are the following functions differentiable at $x = 2$? Why or why not?

(a) $f(x) = \begin{cases} x & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$

(b) $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4x - 4 & \text{if } x \geq 2 \end{cases}$

(c) $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x \geq 2 \end{cases}$

Solution.

(a) To be differentiable at $x = 2$, the limit

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

must exist. In particular, the left-hand and right-hand limits must agree. Since $f(x) = x$ when $x \leq 2$, the left-hand limit is

$$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{(2+h) - (2)}{h} = \lim_{h \to 0^-} \frac{h}{h} = 1$$

Since $f(x) = 3$ when $x > 2$, the right-hand limit is

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{3 - 3}{h} = 0$$

We see that the left-hand and right-hand limits are not the same, so f is *not* differentiable at $x = 2$.

(b) As before, the limit $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$ must exist if we want $f(x)$ to be differentiable at $x = 2$, and so the left and right limits must be the same. The left-hand limit is computed by using x^2 for $f(x)$:

$$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{(2+h)^2 - (2)^2}{h} = \lim_{h \to 0^-} \frac{4h + h^2}{h} = \lim_{h \to 0^-} (4 + h) = 4$$

The right-hand limit is computed by using $4x - 4$ for $f(x)$:

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{[4(2+h) - 4] - [4(2) - 4]}{h} = 4$$

Since the left and right limits are the same, $f(x)$ is differentiable at $x = 2$.

(c) As before, the limit $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$ must exist if we want $f(x)$ to be differentiable at $x = 2$, and so the left and right limits must be the same. The left-hand limit is computed by using x^2 for $f(x)$:

$$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{(2+h)^2 - (2)^2}{h} = \lim_{h \to 0^-} \frac{4h + h^2}{h} = \lim_{h \to 0^-} (4 + h) = 4$$

The right-hand limit is computed by using 4 for $f(x)$:

$$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{4-4}{h} = 0$$

These are not the same, so $f(x)$ is *not* differentiable at $x=2$.

Example 2.9 Compute (using the limit definition) the derivative of the following functions:

(a) $f(x) = 3x^2 + x - 8$

(b) $f(x) = \frac{5}{x-3}$

(c) $f(x) = \frac{1}{x^2}$

Solution.

(a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{[3(x+h)^2 + (x+h) - 8] - [3x^2 + x - 8]}{h} \\
&= \lim_{h\to 0} \frac{6xh + h + h^2}{h} \\
&= \lim_{h\to 0} (6x + 1 + h) \\
&= 6x + 1
\end{aligned}$$

(b)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{5}{(x+h)-3} - \frac{5}{x-3}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{5}{x-3+h} - \frac{5}{x-3}\right) \\
&= \lim_{h\to 0} \frac{5}{h}\left(\frac{x-3-(x-3+h)}{(x-3)(x-3+h)}\right) \\
&= \lim_{h\to 0} \frac{-5h}{h(x-3)(x-3+h)} \\
&= \lim_{h\to 0} \frac{-5}{(x-3)(x-3+h)} \\
&= \frac{-5}{(x-3)^2}
\end{aligned}$$

(c)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(x+h)^2}-\frac{1}{x^2}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right) \\
&= \lim_{h\to 0} \frac{-2xh-h^2}{h\cdot x^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2} \\
&= \frac{-2x}{x^4} \\
&= \frac{-2}{x^3}
\end{aligned}
$$

Example 2.10 Is the function $f(x) = \begin{cases} 2x+1 & \text{if } x<0 \\ x^2+1 & \text{if } x \geq 0 \end{cases}$ differentiable at $x=0$? Why or why not?

Solution. The function will be differentiable if both the left and right limits (as $h \to 0$) of $\dfrac{f(x+h)-f(x)}{h}$ exist and are equal. We calculate the left limit using $2x+1$:

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{[2h+1]-[2(0)+1]}{h} = 2$$

The right limit is calculated by using x^2+1:

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{[h^2+1]-[0^2+1]}{h} = \lim_{h\to 0^+} h = 0$$

Since the left-hand and right-hand limits of $\dfrac{f(0+h)-f(0)}{h}$ are not equal, the function is not differentiable at $x=0$.

Example 2.11 Assuming a function $f(x)$ is differentiable at $x=c$, come up with a general equation for the tangent line to f at c.

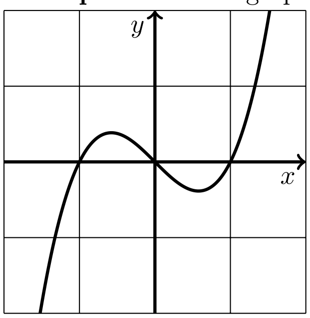
Solution. We know, from **Remark 1.2**, that the slope of the tangent line at $x=c$ is $f'(c)$. We also know that the tangent line goes through the point $(c, f(c))$, which is on the graph of f. Now, using the point-slope formula for a line, we have that the tangent line is described by the equation:

$$y - f(c) = f'(c)(x-c)$$

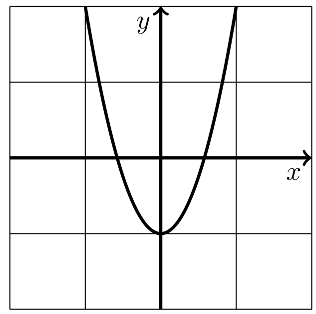
We can solve this for y to get the slope-intercept form:

$$y = f'(c)(x-c) + f(c)$$

Example 2.12 The graph of a function $f(x)$ is shown on the left. Use it to sketch a graph of $f'(x)$.



$f(x)$



$f'(x)$