Example 8.15 Explain why $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 + \sin x & \text{if } x > 0 \end{cases}$ is discontinuous at $x = 0$. Write the largest interval on which $f(x)$ is continuous.

Solution. The left and right-hand limits (as $x \to 0$) are both equal to 1, but $f(0) = 0$.
Since $\lim_{x\to 0} f(x) \neq f(0)$, the function is not continuous (by definition).

Example 8.16 Locate the discontinuities of the function $f(x) = \dfrac{2}{1 - \sin(x)}$.

Solution. Since $\frac{2}{x}$ is continuous everywhere except at $x = 0$, $f(x)$ will be continuous everywhere that $1 - \sin(x)$ is not equal to zero (by **Theorem 8.7**). If $1 - \sin(x) = 0$, this just means that $\sin(x) = 1$. This happens if $x = \dfrac{\pi}{2}$ (or this plus a multiple of 2π). So the points at which $f(x)$ is discontinuous consists of the numbers:

$$x = \frac{\pi}{2} + 2\pi k$$

for all integers k.

Example 8.17 Show that $|x|$ is continuous everywhere.

Solution. If $x > 0$, then $|x| = x$. Since $f(x) = x$ is a polynomial, it is continuous (by **Theorem 8.3**).
If $x < 0$, then $|x| = -x$. Since $f(x) = -x$ is a polynomial, it is continuous.
So we just have to check that $|x|$ is continuous at $x = 0$. That is, we have to check that:

$$\lim_{x\to 0} |x| = |0| = 0$$

But since $|x| = x$ when $x > 0$, we see that $\lim_{x\to 0^+} = 0$, and since $|x| = -x$ when $x < 0$, we see that $\lim_{x\to 0^-} = 0$. Since the left and right limits agree, the two-sided limit exists:

$$\lim_{x\to 0} |x| = 0$$

This shows that $|x|$ is continuous at zero.

Example 8.19 For what constant c is the function f continuous everywhere?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

Solution. The function is certainly continuous on $(-\infty, 2)$ and on $(2, \infty)$, since it is equal to a polynomial on these intervals. We just need to check at $x = 2$. The limit as we approach 2 from the left is:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} cx^2 + 2x = c(2)^2 + 2(2) = 4c + 4$$

The limit as we approach 2 from the right is:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^3 - cx = (2)^3 - c(2) = 8 - 2c$$

In order for $f(x)$ to be continuous at $x = 2$, we'd need these two one-sided limits to be equal. We can solve algebraically to see what value of c will make this happen:

$$\begin{aligned} 4c + 4 &= 8 - 2c \\ 6c &= 4 \\ c &= \frac{2}{3} \end{aligned}$$

Example 8.20 Prove that the equation $\sin(x) = x^2 - x$ has at least one solution in the interval $(1, 2)$.

Solution. Let $f(x) = \sin(x) + x - x^2$. Then a solution to the equation $\sin(x) = x^2 - x$ is also a solution to the equation $f(x) = 0$. We know the function $f(x)$ is continuous (since $\sin(x)$, x, and x^2 are all continuous). If we evaluate f at $x = 1$ and $x = 2$, we get $f(1) = \sin(1)$ and $f(2) = \sin(2) - 2$. Certainly $\sin(2) - 2$ is negative (since the range of the sine function is $[-1, 1]$). Converting 1 radian to degrees, we see that 1 radian is $\frac{180}{\pi}$ degrees, and so $\sin(1)$ will be positive. Then we have the inequality:

$$\sin(2) \leq 0 \leq \sin(1)$$

Then by the **Intermediate Value Theorem**, the equation $f(x) = 0$ has a solution in the interval $[1, 2]$.

Example 8.21 Prove that the equation $\cos(x) = x^3$ has at least one solution. What interval is it in?

Solution. Let $f(x) = \cos(x) - x^3$. Then solutions to $\cos(x) = x^3$ will also be solutions to $f(x) = 0$. Notice that $f(0) = 1$. So if we can find a number c so that $f(c) \leq 0$, then we'd have that $f(c) \leq 0 \leq f(0)$, and so the **Intermediate Value Theorem** will tell us that $f(x) = 0$ has a solution in between 0 and c. In fact, $c = 1$ will work, since $(1)^3 = 1$, and $\cos(1) \leq 1$. So the solution to $\cos(x) = x^3$ is in the interval $[0, 1]$.