Example 7.10 If $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for $x \geq 0$, find $\lim_{x \to 4} f(x)$.

Solution: Since $\lim_{x \to 4} 4x - 9$ and $\lim_{x \to 4}(x^2 - 4x + 7)$ are both equal to 7, and we already know that $4x - 9 \leq f(x) \leq x^2 - 4x + 7$, we can use the **Squeeze Theorem** to say that $\lim_{x \to 4} f(x) = 7$, as well.

Example 7.11 Find $\lim_{x \to 0} x^4 \cos\left(-\frac{1}{x}\right)$.

Since $\cos\left(-\frac{1}{x}\right)$ is always between -1 and 1, we know that

$$-x^4 \leq x^4 \cos\left(-\frac{1}{x}\right) \leq x^4$$

Since $\lim_{x \to 0} x^4$ and $\lim_{x \to 0}(-x^4)$ are both zero, so is $\lim_{x \to 0} x^4 \cos\left(-\frac{1}{x}\right)$, by the **Squeeze Theorem**.

Example 7.12 Prove that $\lim_{x \to 2}(5 - x) = 3$, using the formal definition of a limit.

We have to show that *for any* positive number ε, we can find another positive number δ, so that $|(5-x)-3| < \varepsilon$ as long as $|x - 2| < \delta$. We do this by taking the first inequality, and trying to solve it:

$$\begin{aligned} |(5 - x) - 3| &< \varepsilon \\ |2 - x| &< \varepsilon \\ |x - 2| &< \varepsilon \end{aligned}$$

This tells us that as long as x is within a distance ε of 2, $5 - x$ will be within a distance ε of 3. So we can use $\delta = \varepsilon$ in the proof.

Example 7.13 Prove that $\lim_{x \to a} c = c$ and that $\lim_{x \to a} x = a$ using the definition of the limit.

First let's prove that $\lim_{x \to a} c = c$. In other words, the limit of a constant is the constant, no matter which point we are taking the limit to. We need to show that *for any* $\varepsilon > 0$, we can find a $\delta > 0$ so that $|c - c| < \varepsilon$. But since $c - c = 0$, this just says that $0 < \varepsilon$, which we already know. This is always true, without imposing any restrictions on x. So in fact, *any* choice of δ would work in this case (even if it's not small).

Next let's show that $\lim_{x \to a} x = a$. We need to show that *for any* $\varepsilon > 0$, there is a $\delta > 0$ so that $|x - a| < \varepsilon$ whenever $|x - a| < \delta$. But we see that we can obviously just pick $\delta = \varepsilon$, and it will work.