# 5 The Substitution Rule

**Theorem 5.1.** If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

**Remark 5.2.** The Substitution Rule says essentially that is is permissible to operate with $dx$ and $du$ after the integral signs as if they were differentials.

**Theorem 5.3.** If $g'(x)$ is continuous on $[a, b]$ and $f$ is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

**Remark 5.4.** This change in the limits of integration is annoying and somewhat unnecessary right now (so long as you change your variable back) however in Calc II once you start doing trig substitutions it becomes extremely useful. WeBWorK will force you to practice changing the limits of integration so we will practice here in class too.

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**Definition(s) 5.5.** Recall again that $f$ is called **even** if $f(-x) = f(x)$ and called **odd** if $f(-x) = -f(x)$.

**Theorem 5.6.** Suppose $f$ is continuous on $[-a, a]$ then:

(a) If $f$ is even, then $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$

(b) If $f$ is odd, then $\int_{-a}^{a} f(x)\ dx = 0$

**Remark 5.7.** This make logical sense if you draw out a picture and sometimes get used for tricky multiple choice / fill-in-the-blank questions.

**Example 5.8** (Instructor)**.** Evaluate the following indefinite integrals:

(a) $\displaystyle\int x^2\sqrt{x^3+1}\ dx$

(b) $\displaystyle\int \frac{dt}{(1-3t)^5}$

(c) $\displaystyle\int \sec^3 x \tan x\ dx$

(d) $\displaystyle\int \frac{\sin t}{\cos^2(\cos(t))}\ dt$

**Example 5.9** (Instructor)**.** Evaluate the following definite integrals:

(a) $\displaystyle\int_0^3 \frac{x}{\sqrt{1+5x}}\ dx$

(b) $\displaystyle\int_{-\pi/4}^{\pi/4} (1+x+x^2\tan x)\ dx$

**Example 5.10** (Instructor)**.** Evaluate the following definite integrals by changing the limits of integration appropriately:

(a) $\displaystyle\int_0^1 \sqrt[3]{1+7x}\ dx$

(b) $\displaystyle\int_0^{\sqrt{\pi}} x\cos(x^2)\ dx$

**Example 5.11** (Student)**.** Evaluate the following indefinite integrals:

(WW2) $\displaystyle\int \frac{x^3}{\sqrt{5+9x^4}}\ dx$

(WW6) $\displaystyle\int \frac{9\sin(\sqrt{x})}{\sqrt{x}}\ dx$

(WW8) $\displaystyle\int \frac{1}{x^2}\sin\left(\frac{3}{x}\right)\cos\left(\frac{3}{x}\right)\ dx$

(WW11) $\displaystyle\int 5x\sqrt{x-4}\ dx$

**Example 5.12** (Student)**.** Evaluate the following definite integrals by changing the limits of integration appropriately:

(WW12) $\displaystyle\int_{-4}^{2} \frac{1}{\sqrt{9-2x}}\ dx$

(WW13) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos(z)}{\sin^{5/2}(z)}\ dz$

**Example 5.13** (Student)**.** Evaluate the following definite integrals:

(WW15) $\displaystyle\int_{-\pi/16}^{\pi/16} (4x+\sin(8x))\ dx$

(WW16) $\displaystyle\int_{-4}^{4} \frac{t^3}{13+2t^2}\ dt$